The Dissertation Committee for Jacob Benjamin Glenn-Levin
certifies that this is the approved version of the following dissertation:

Incompressible Boussinesq Equations and Spaces of Borderline Besov Type

Committee:

Mikhail Vishik, Supervisor

Irene Gamba

Phil Morrison

Nataša Pavlović

Yen-Hsi Richard Tsai

Alexis Vasseur

Incompressible Boussinesq Equations and Spaces of Borderline Besov Type

by

Jacob Benjamin Glenn-Levin, B.A.; M.A.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2012

Dedicated to my parents, Jeff and Sara.

Acknowledgments

First, I wish to thank my advisor, Misha Vishik, for his help and guidance during my years in candidacy and for his invaluable advice and encouragement in my research. It has been a joy to learn from and work with him, and I will be forever grateful for both the math he and I discussed in our weekly meetings and the math that he had the foresight to suggest I learn on my own instead - that distinction has made me a better mathematician.

I wish to thank my fellow grad students who I spent time with during my years at University of Texas at Austin, especially Oğuz Erdin, Dan Blazevski, Tim Blass, Mio Alter and Sean Bowman. I am extremely thankful for both the mathematics we discussed over the years, as well as the friendship we shared. It is of vital importance to know when to push your work harder and when to take a break to collect your thoughts and approach things from a new perspective, and I am lucky to have learned both from such a good group of friends and colleagues. In addition, I wish to thank John Garza, with whom I shared an office during my first two years of graduate studies. Without his willingness to discuss mathematics with a wet-behind-the-ears graduate student, I would not have succeeded in the difficult early years of the doctoral program.

The staff of the Mathematics Department at University of Texas at

Austin deserve a whole-hearted thanks for all of the help they have provided me over the years. I thank Sandra Catlett and her predecessor Nancy Lamm for helping guide me through the graduate student process. Without their help, I and so many other graduate students would have been utterly lost in the mazes of graduate school procedure. I wish to express my gratitude to Dan Knopf and Bruce Palka, who served as faculty advisors during my time as a graduate student. I admire Dan's unflagging enthusiasm for every graduate student's work and progress, and without his support my years here would surely not have been as enriching. I thank Dr. Palka deeply for the kindness he showed in inviting an (at the time) unadmitted student to visit the department and speak to faculty and other graduate students. He was the first of many people in the Mathematics Department who believed in my potential and without whom I would not have been able to complete my doctorate.

Special thanks go to my family - my parents, Jeff and Sara, and my sister Naomi. They have been endless sources of encouragement for all of my life, and I will be forever grateful for their love, patience, kindness and for their indefatigable support of my work. As well, I wish to thank my Grandfather Abe, who not only showed me at a very young age that education should be treated as a lifelong pursuit, but also for his undying belief (and frequent reminders) that I was capable of whatever I aspired to.

Finally, I wish to thank Karin Mente, who has been my greatest support and most treasured confidant over the last few years. I have taken her endless enthusiasm for life and her willingness to take on new challenges more to heart

than I could possibly indicate in words. Her patience and clarity of thought have helped me through many frustrating days, and my life has been markedly better since we met. I am and will continue to be ever grateful for the support she provides.

Incompressible Boussinesq Equations and Spaces of Borderline Besov Type

Publication No. ____________________

Jacob Benjamin Glenn-Levin, Ph.D.
The University of Texas at Austin, 2012

Supervisor: Mikhail Vishik

The Boussinesq approximation is a set of fluids equations utilized in the atmospheric and oceanographic sciences. They may be thought of as inhomogeneous, incompressible Euler or Navier-Stokes equations, where the inhomogeneous term is a scalar quantity, typically representing density or temperature, governed by a convection-diffusion equation.

In this thesis, we prove local-in-time existence and uniqueness of an inviscid Boussinesq system. Furthermore, we show that under stronger assumptions, the local-in-time results can be extended to global-in-time existence and uniqueness as well. We assume the density equation contains nonzero diffusion and that our initial vorticity and density belong to a space of borderline Besov-type. We use paradifferential calculus and properties of the Besov-type spaces to control the growth of vorticity via an a priori estimate on the growth of density. This result is motivated by work of M. Vishik demonstrating local-

in-time existence and uniqueness for 2D Euler equations in borderline Besov-type spaces, and by work of R. Danchin and M. Paicu showing the global well-posedness of the 2D Boussinesq system with initial data in critical Besov and L^p-spaces.

Table of Contents

Chapter 1

Introduction

In this thesis, we consider the two dimensional Boussinesq system given by

$$(B_{\kappa,\nu}) \begin{cases} \partial_t u + (u, \nabla)u - \nu\Delta u + \nabla P = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \\ \partial_t \rho + (u, \nabla)\rho = \kappa\Delta\rho \\ \text{div } u = 0 \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x). \end{cases}$$

Where $u = (u_1, u_2)$ is the velocity field, ρ is the scalar density or temperature and P is the pressure. This system has relevance in the study of atmospheric and oceanographic dynamics and turbulence where rotation or stratification occurs (cf. [20]). Of mathematical interest, note that for $\rho \equiv 0$ we recover the incompressible Navier-Stokes and Euler Equations (for $\nu > 0$ and $\nu = 0$, respectively). Furthermore, the system $(B_{0,0})$ has vortex stretching similar to that of the of 3D axisymmetric flow, thus providing a model under which to approach the formation of finite time singularities (see [21] for discussion of this relationship).

The structure of this dissertation is as follows: in the remainder of this chapter, I provide some formal background for the Boussinesq equations, and discuss some of the motivation involved in the approximations it utilizes. Next, we discuss some previous results that deal with the Boussinesq system and the function spaces we will study in the remainder of this work. Finally, the chapter concludes with a brief discussion of notational assumptions that will be made throughout the rest of the thesis.

In the second chapter, I present some of the background necessary to formulate and prove the main results of the thesis. I begin by describing some classical results in the realm of mathematical fluid mechanics. Following that, I introduce the function spaces to be utilized in this thesis, the Besov spaces and the related B_Γ spaces. In addition, I describe several embeddings and equivalences between these spaces and more common function spaces. Finally, chapter two ends with a discussion of some of the propositions and lemmas I will make use of to prove the results of chapter three, such as Bernstein's inequality, commutator estimates in the spirit of Bahouri and Chemin's paper [1], and an inequality of Chemin's concerning the convection-diffusion equation.

In chapter three, I prove the existence and uniqueness of solutions to the two-dimensional Boussinesq equations under the assumption that the initial data belongs to B_Γ, along with some membership in L^p-spaces. The structure of the proof is roughly as follows - I show that for positive time, the growth of vorticity and density is bounded a priori by a locally integrable function. Using this fact, I next prove uniqueness by showing that for two solutions, the $B^0_{\infty,1}$ norm of their difference is bounded above by a monotonically increasing, nonnegative, absolutely continuous function. Using some results from ODE theory, I prove by contradiction that that function is identically zero on a positive time interval, hence uniqueness must hold at least locally in time and, under the proper assumptions, globally in time as well. Note that the uniqueness result holds for a weaker choice of function $\Gamma(\alpha)$ than that described in section 2.2. Finally, using the existence of solutions in Sobolev spaces shown by D. Chae in [6], I prove the existence of solutions to $(B_{\kappa,0})$ given initial data in $B_\Gamma(\mathbb{R}^2)$ using an extension of the argument used to show uniqueness, along with a utilization of the dual-space of B_Γ to show that said solutions are weak-* continuous in time with values in a related space B_{Γ_1}.

Lastly, in the appendix I provide some details and proofs of the properties of the B_Γ spaces discussed by M. Vishik in his paper [31] proving local-in-time existence for the two-dimensional incompressible Euler system with initial data in B_Γ. Specifically, I discuss how a volume-preserving homeomorphism acts on a function in B_Γ. This property is key to the results in this thesis, and in the appendix I provide more details of Vishik's results than are useful in the exposition of chapter three.

1.1 Motivation for the Boussinesq equations

The Boussinesq equations arise from making certain simplifying assumptions when studying fluid problems involving thermal convection. These equations were first utilized in 1903 by J. Boussinesq in [4], where he developed them to study convection in a narrow layer of compressible fluid. They can best be summarized as follows (for a more detailed discussion and development of the Boussinesq equations, see [18, 26]):

1. Fluctuations in density which appear with the advent of motion result principally from thermal effects.

2. In the conservation of momentum and mass equations, density variations may be neglected except when they are coupled to the gravitational acceleration in the buoyancy force.

For a compressible, inviscid fluid system in two dimensions, the equations of conservation of mass and momentum are given, respectively, by

$$\begin{aligned}\frac{1}{\rho}\frac{D\rho}{Dt} + \text{div } u = 0\\ \rho\frac{Du}{Dt} = -\nabla P - g\rho e_2 + \nu\Delta u,\end{aligned}$$

where ρ is the density, P the pressure, g the gravitational constant, e_2 is the vector $(0, 1)^\top$ and $\frac{D}{Dt}$ is the material derivative

$$\frac{Df}{Dt} = \partial_t f + (u, \nabla)f.$$

The second assumption above transforms this system into

$$\text{div } u = 0 \tag{1.1}$$

$$\frac{Du}{Dt} = -\nabla P - g\rho e_2 + \nu\Delta u \tag{1.2}$$

Note that the assumption made in simplifying the mass equation is not that the first term is identically zero, but that

$$\frac{1}{\rho}\frac{D\rho}{Dt} \ll \text{div } u.$$

If one normalizes the gravitational constant to be $g = 1$, we arrive at a system which appears as an inhomogeneous, incompressible Navier-Stokes system with viscosity ν and right-hand side given by a buoyancy term $-\rho e_2$. Clearly, assuming $\nu \equiv 0$ gives the related Euler system.

Next, consider the effects of the first assumption. For an inviscid system, a simplified equation governing the evolution of temperature is given by:

$$\rho \frac{D\theta}{Dt} + P(\text{div } u) = \kappa \Delta \theta,$$

where κ is the heat diffusivity constant. If one assumes that fluctuations in density are primarily based upon θ, then we may think of ρ as

$$\frac{\rho - \rho_0}{\rho_0} \sim \alpha_0(\theta - \theta_0),$$

where θ_0, ρ_0 are the (spatial) averages of temperature and density, and α_0 is a constant. While the Boussinesq approximation assumes the conservation of mass equation is given by div $u = 0$, the term $P(\text{div } u)$ in the temperature equation is not negligible compared to the other terms. Utilizing the compressible version of the conservation of mass equation and the ideal gas law, however, one can show that

$$P(\text{div } u) = \frac{P}{\rho} \frac{D\rho}{Dt} \sim \rho \frac{D\theta}{Dt}.$$

This implies that the evolution of temperature can be written as

$$\frac{D\theta}{Dt} = \kappa \Delta \theta, \tag{1.3}$$

after possibly adjusting the value of κ, which yields the final equation in the system $(B_{\kappa,0})$ listed above (albeit with θ in place of ρ).

1.2 Previous results

From an existence and regularity perspective, much progress has been made recently on the Boussinesq equations. When both κ and ν are strictly positive, Ukhovskii and Yudovich [29] showed in the mid-1960s that such a system has a unique steady state solution. More recently, [5, 16] give global existence of smooth solutions using standard energy method arguments. In the case of $\kappa \equiv 0, \nu = 0$, local well-posedness as well as a blow-up criterion similar to well-known result of Beale, Kato and Majda [2] has been shown in a number of function spaces [7, 8, 19]. While there has been some numerical study of finite time singularity for $(B_{0,0})$, the results remain inconclusive [14, 25]. For $\kappa \equiv 0$, $\nu > 0$, [27] shows global existence of a solution to $(B_{0,\nu})$ for non-decaying initial data: $(u_0, \rho_0) \in \mathrm{L}^\infty(\mathbb{R}^2) \times \dot{B}^0_{\infty,1}(\mathbb{R}^2)$. Recently, R. Danchin and M. Paicu in [13] prove a Yudovich-type result in $\mathbb{R}^2$ for $(B_{\kappa,0})$ under the assumption that $u_0 \in \mathrm{L}^2$, $\omega_0 \in \mathrm{L}^r \cap \mathrm{L}^\infty$ $(r \geq 2)$ and $\rho_0 \in \mathrm{L}^2 \cap B^{-1}_{\infty,1}$.

In this thesis, I study the case of $\nu = 0, \kappa > 0$. Recall that in the celebrated paper [32], V. Yudovich proves the existence and uniqueness of a weak solution to the 2D incompressible Euler Equations,

$$
\text{(E)} \begin{cases} \partial_t u + (u, \nabla) u + \nabla P = 0 \\ \text{div } u = 0 \\ u(x, 0) = u_0(x), \end{cases}
$$

for initial vorticity $\omega_0 = \text{curl } u_0 \in \mathrm{L}^\infty(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for any $p \in (1, \infty)$. In [33], Yudovich extends the uniqueness portion of this result to include vorticity which is unbounded but whose L^p-norm grows like $\log(p)$ as $t \to \infty$ - this corresponds to double log-type singularities for vorticity. In more recent years, Yudovich's existence result has been extended for initial vorticity in more general function spaces including, among others, Besov spaces. Recall that the (inhomogeneous) Besov Space, $B^s_{p,q}$, can be characterized as the set of all $f \in \mathcal{S}'$ such that

$$\left(\sum_{j=-1}^{\infty} 2^{jqs} \left\| \Delta_j f \right\|_p^q \right)^{\frac{1}{q}} < \infty,$$

where $s \in \mathbb{R}$, $p, q \in [1, \infty]$ and Δ_j is the Littlewood-Paley operator (which is defined precisely in chapter two).

In [31], M. Vishik proves the existence and uniqueness of a local-in-time solution to the incompressible 2D Euler Equations with initial data in a critical Besov-type Space (critical in the sense $s = n/p$ - see [28] for a thorough discussion of critical, sub-critical and super-critical Besov Spaces). He introduces the B_Γ spaces, based on the $B^0_{\infty,1}$-norm, such that

$$B_\Gamma = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \middle| \sum_{j=-1}^{N} \left\| \Delta_j f \right\|_\infty = O(\Gamma(N)) \right\},$$

and shows that for initial data in such a space, sufficient control on the growth of $\Gamma(\alpha)$ as $\alpha \to \infty$ gives existence in a slightly weaker B_Γ-type space for positive time. In [12] E. Cozzi and J. Kelliher prove that the vanishing viscosity limit in $\mathbb{R}^2$ holds for initial data in $B_\Gamma \cap \mathrm{L}^2$ and extend slightly the class of functions $\Gamma(\alpha)$ for which global existence and uniqueness of the Euler equations can be found.

Given these above-mentioned results, I seek to prove the local and global well-posedness of $(B_{\kappa,0})$ while relaxing the boundedness constraints of

Danchin and Paicu on the initial data. Inspired by the results of Vishik on the Euler equations, I replace their initial data assumption of L^∞ with the more general membership in B_Γ for suitable choice of Γ. In addition, this thesis has the secondary distinction of extending Vishik's result to the Boussinesq equations, a natural generalization of the Euler equations.

1.3 Regarding Notation

In this thesis, I adopt the convention that the constant C is a generic constant, which may differ from line-to-line of a calculation. When dependencies are relevant, I write $C(\cdot)$ or $C_{(\cdot)}$ to clearly indicate what the constant depends upon. If multiple, distinct constants are required, they will be denoted as either C_0, C_1, C_2, etc. or C, c to distinguish them from each other.

With respect to norms, I adopt the convention that L^p norms are denoted as $\|\cdot\|_p$, and that norms with respect to B_Γ space are denoted $\|\cdot\|_\Gamma$. All other norms will have the appropriate function space listed in subscript unless explicitly stated otherwise. Furthermore, for two function spaces $\mathcal{F}, \mathcal{G}$ I use the convention that $\|\cdot\|_{\mathcal{F}\cap\mathcal{G}} := \max\{\|\cdot\|_{\mathcal{F}}, \|\cdot\|_{\mathcal{G}}\}$. Finally, I use the notation $a \simeq b$ to indicate that there exists a constant C, independent of a and b, such that $C^{-1}a \leq b \leq Ca$.

Chapter 2

Background

In this section, I review some classical results from fluid dynamics and partial differential equations, define relevant function spaces and state some important lemmas I will make use of in the remainder of this thesis.

2.1 Classical Results

First, I mention Abel's Lemma, also known as summation-by-parts, a formula I will make occasional use of in proving the results of chapter three. The proof is a straightforward induction argument, and is hence omitted:

Proposition 2.1.1. *Let $\{a_i\}$ and $\{b_i\}$ be sequences of real numbers, and define A_N to be the sum of the first N terms of $\{a_i\}$, $A_N = \sum_{i=0}^{N} a_i$. Then for $N \geq 0$, the following identity holds:*

$$\sum_{i=0}^{N} a_i b_i = A_N b_N + \sum_{i=0}^{N-1} A_i (b_i - b_{i+1})$$

Next, I introduce the Biot-Savart law, which demonstrates the relationship between vorticity and velocity. While the result is classical, I present here a version adapted from [9]:

Theorem 2.1.2. *Let $\omega \in \mathrm{L}^p$ for $p < d$. If $q > \frac{pd}{p-d}$, then there exists a unique divergence-free velocity field in $\mathrm{L}^p + \mathrm{L}^q$ with curl $v = \omega$. Furthermore, if $\mathrm{E_d}$ denotes the fundamental solution to the Laplacian in dimension d, then one can reconstruct v from ω, namely*

$$v^i(x) = \sum_k \partial_k E_d * \omega_k.$$

In order to prove the Biot-Savart law, I make use of the following lemma:

Lemma 2.1.3. *Two vector fields whose coefficients are tempered distributions and whose divergence and vorticity are equal, are equal up to a vector field with harmonic polynomials as coefficients.*

Proof. (Lemma) Let $\Omega(v)^i_j = \partial_j v^i - \partial_i v^j$, then one can write $\partial_i v^j = \partial_j v^i + \Omega(v)^i_j$ for $i \neq j$. Taking the ith derivative of both sides of this equation and summing from $i = 1$ to d gives:

$$\Delta v^i = \partial_j \text{div } v + \sum_i \partial_i \Omega(v)^i_j.$$

If $v, \tilde{v}$ are the two vector fields whose divergence and vorticity are equal, then we have $\Delta(v - \tilde{v}) = 0$, which implies that $v, \tilde{v}$ differ by a harmonic polynomial. $\square$

Proof. (Biot-Savart law) Let $\Omega(v) = (\Omega^i_j(v))_{1 \leq i,j \leq n}$ be the matrix associated with the curl of a velocity field v. (In two dimensions, $\Omega(v)$ is given by the usual $\omega(v) = \partial_1 v^2 - \partial_2 v^1$.) For $1 \leq i \leq d$, set $\tilde{v}^i = \sum_k \partial_k E_d * \Omega^i_k(v)$. I wish to show that $\tilde{v} = v$. First, I show that $\Omega(\tilde{v}) = \Omega(v)$. For fixed i, j,

$$\begin{aligned}
\Omega(\tilde{v})^i_j &= \sum_k \left\{\partial_k\partial_j E_d * (\partial_k v^i - \partial_i v^k) - \partial_k\partial_i E_d * (\partial_k v^j - \partial_j v^k)\right\} \\
&= \sum_k \left\{\partial_k^2 E_d * \partial d_j v^i - \partial_j E_d * \partial_i\partial_k v^k - \partial_k^2 E_d * \partial_i v^j + \partial_j E_d * \partial_i \partial_k v^k\right\} \\
&= \sum_k \left\{\partial_k^2 E_d * (\partial_j v^i - \partial_i v^j)\right\} = \Omega(v)^i_j.
\end{aligned}$$

Next, taking the divergence of $\tilde{v}$, one has

$$\operatorname{div}\, \tilde{v} = \sum_{i,k} \partial_i\partial_k E_d * (\partial_k v^i - \partial_i v^k) = 0.$$

Note that since ω belongs to L^p and $\partial_i E_d$ belongs to $\mathrm{L}^p + \mathrm{L}^s$ for $s > d$, $\tilde{v}$ belongs to $\mathrm{L}^p + \mathrm{L}^q$ by Young's inequality. By lemma 2.1.3, v and $\tilde{v}$ differ by a harmonic polynomial, and the proof is complete. $\square$

Given a vector field $v(x,t)$, define the flow map associated with that vector field to be the function $X(x,t)$ which solves:

$$\begin{cases} \dfrac{dX}{dt}(\alpha, t) = v(X(\alpha,t),t) \\ X(\alpha, 0) = 0. \end{cases}$$

In other words, the flow map $X(\alpha, t)$ tells the position of a particle after it has travelled according to the velocity field $v(x,t)$ for t units of time from an original position $\alpha \in \mathbb{R}^d$ at $t = 0$. When the vector field is divergence-free, this flow map is volume-preserving. To wit, one has the following classical result, this version of which can be found in [21]:

Proposition 2.1.4. *For a smooth flow map $X(x,t)$ and its associated velocity field $v(x,t)$, the following are equivalent:*

- *The flow map is volume-preserving, i.e. for all $\Omega \subset \mathbb{R}^n$ and $t > 0$, vol $X(\Omega,t) = $ vol (Ω).*
- *The velocity field is divergence free, i.e. div $v = 0$.*

Proof. Let Ω be an arbitrary fixed subset of $\mathbb{R}^n$. Then X is a volume-preserving flow map if and only if one has

$$\frac{\mathrm{d}}{\mathrm{d}t}[\text{vol } X(\Omega,t)] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{X(\Omega,t)} \mathrm{d}x\right] = 0.$$

One can rewrite this integral as $\int_\Omega J(\alpha,t)\mathrm{d}\alpha$, where $J(\alpha,t)$ is the Jacobian given by

$$J(\alpha,t) = \det\,(\nabla X(\alpha,t)), \quad \frac{\partial J}{\partial t} = (\text{div } v)|_{(X(\alpha,t),t)} J(\alpha,t).$$

This implies that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J(\alpha,t)\mathrm{d}\alpha \\ &= \int_\Omega \frac{\partial J}{\partial t}(\alpha,t)\mathrm{d}\alpha \\ &= \int_\Omega (\text{div } v)|_{(X(\alpha,t),t)} J(\alpha,t)\mathrm{d}\alpha, \end{aligned}$$

which is true if and only if div $v = 0$. □

Notationally, I write $X(x,t;\tau)$ to be the flow map such that $X(x,0;\tau) = x(\tau)$, i.e., $X(\cdot,t;\tau)$ is the flow map given by X_0 after having evolved for time $\tau > 0$.

Next, I state the Osgood Uniqueness theorem, a result from ODE theory that will be useful in the proofs of chapter three. For a proof of this result, see section 5.2 of Chemin's book [9]:

Theorem 2.1.5. *Let E be a Banach space, Ω an open set of E, I an open interval of $\mathbb{R}$ and (t_0, x_0) an element of $I \times \Omega$. Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, increasing, strictly positive function such that $\mu(0) = 0$, and suppose that*

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty.$$

Let $\mathcal{C}_\mu(X, E)$ denote the set of bounded functions $u : X \to E$ such that $\|u(x) - u(y)\|_E \leq C\mu(d(x, y))$. Consider a function $F \in L^1_{loc}(I; \mathcal{C}_\mu(X, E))$, then there exists and interval $J \subset I$ containing t_0 such that the equation

$$x(t) = x_0 + \int_{t_0}^t F(s, x(s))\, ds$$

has a unique continuous solution defined on the interval J.

Finally, I make use of this classical result found in chapter 3.1 of [9] which details the relationship between the gradient of a divergence-free velocity field and the vorticity of that field, $\omega = \operatorname{curl} v$:

Proposition 2.1.6. *Let v be a divergence-free vector field whose gradient belongs to L^p. Then for $p \in (1, \infty)$, we have*

$$\|\nabla v\|_p \leq C \frac{p^2}{p-1} \|\omega\|_p .$$

2.2 Function spaces

In order to more easily define the function spaces to be used in the following chapters, I first introduce the notion of the Littlewood-Paley decomposition. Let the hat operator denote the usual Fourier transform, and let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that

$$\operatorname{supp} \hat{\Phi} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq 1\}, \quad |\hat{\Phi}(\xi)| \geq C > 0 \text{ on } \left\{|\xi| \leq \frac{5}{6}\right\}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function such that

$$\operatorname{supp} \hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2\}, \quad |\hat{\varphi}(\xi)| \geq C > 0 \text{ on } \left\{\frac{3}{5} \leq |\xi| \leq \frac{5}{3}\right\}.$$

Set $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ for $j \in \mathbb{Z}$ (i.e., $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$). Based on this choice of Φ and φ, the Littlewood-Paley operators are given by:

Definition 2.2.1. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\Delta_{-1} f = \hat{\Phi}\left(\frac{1}{i}\frac{\partial}{\partial x}\right) f = \mathcal{F}^{-1}(\hat{\Phi} \cdot \hat{f}) = \Phi \star f, \\
&\Delta_j f = \hat{\varphi}_j\left(\frac{1}{i}\frac{\partial}{\partial x}\right) f = \mathcal{F}^{-1}(\hat{\varphi} \cdot \hat{f}) = \varphi_j \star f \quad \text{for j} \geq 0, \\
&\Delta_j f = 0 \quad \text{for j} \leq -2 \\
&S_j f = \sum_{k=-1}^{j} \Delta_k f.
\end{aligned}$$

From the above, I define the Littlewood-Paley decomposition of $f \in \mathcal{S}'$ as

$$f = \sum_{j=-1}^{\infty} \Delta_j f.$$

With this decomposition in mind, one can define the (inhomogeneous) Besov space $B^s_{p,q}(\mathbb{R}^n)$ as follows:

Definition 2.2.2. Let $s < \infty$, and $1 \leq p, q \leq \infty$. For $q < \infty$, define $B^s_{p,q}(\mathbb{R}^n)$ to be the set of all $f \in \mathcal{S}'$ such that

$$\left(\sum_{j=-1}^{\infty} 2^{jqs} \left\| \Delta_j f \right\|_p^q \right)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, The Besov space $B^s_{p,\infty}$ is the set of all $f \in \mathcal{S}'$ such that

$$\sup_{j \geq -1} 2^{js} \left\| \Delta_j f \right\|_p < \infty.$$

In this thesis, I will be mainly focused on the Besov spaces $B^0_{p,1}$ for $p \in [1, \infty]$. To define the B_Γ spaces, based on the definition of $B^0_{\infty,1}$, one has the following:

Definition 2.2.3. Let $\Gamma : \mathbb{R} \to [1, \infty)$. The space B_Γ is the set of all $f \in \mathcal{S}'$ such that for any $N \geq -1$:

$$\sum_{j=-1}^{N} \left\| \Delta_j f \right\|_\infty \leq C\Gamma(N), \quad \text{where } \left\| f \right\|_\Gamma = \sup_{N \geq -1} (\Gamma(N))^{-1} \sum_{j=-1}^{N} \left\| \Delta_j f \right\|_\infty .$$

Remark 2.2.1. Note that for Γ such that $\Gamma(\alpha) = C\alpha$ when $\alpha \geq 1$, $\Gamma(\alpha) = 1$ otherwise, one has $\mathrm{L}^\infty \subset B_\Gamma$, thus the two spaces are comparable for such a choice of Γ.

For the purposes of this work, let $\Gamma : \mathbb{R} \to [1, \infty)$ satisfy the following conditions:

i. $\Gamma(\alpha) = 1$ for $\alpha \in (-\infty, -1]$, $\lim_{\alpha \to \infty} \Gamma(\alpha) = \infty$

ii. There exists a constant $C > 0$ such that $\Gamma(\beta) \simeq \Gamma(\alpha)$ for $\alpha, \beta \in [-1, \infty), |\alpha - \beta| \leq 1$.

iii. There is a constant $C > 0$ such that for $\alpha \in [-1, \infty)$,

$$C2^{-\alpha}\Gamma(\alpha) \geq \int_{\alpha}^{\infty} 2^{-\xi}\Gamma(\xi)\mathrm{d}\xi.$$

Define $\Gamma_1(\alpha) = (\alpha + 2)\Gamma(\alpha)$ for $\alpha \geq -1$, $\Gamma_1(\alpha) = 1$ otherwise and assume:

iv. Γ_1 satisfies (iii),

v. Γ_1 is convex,

vi. $\int_1^\infty (\Gamma_1(\alpha))^{-1}\mathrm{d}\alpha = \infty$.

Note that a function $\Gamma(\alpha)$ which is constant for $\alpha \leq 0$ and which grows like $\log^\beta(\alpha)$, $\beta \in (0, 1]$, for positive α satisfies the above assumptions.

Lastly, I define J.M. Bony's paraproduct decomposition, introduced in [3], which will prove a vital tool in several results in chapter three of this thesis:

Definition 2.2.4. Let $u, v \in \mathcal{S}'$ be two tempered distributions. I write each with respect to their Littlewood-Paley decomposition as

$$u = \sum_{i \geq -1} \Delta_i u, \text{ and } v = \sum_{j \geq -1} \Delta_j v.$$

Then formally their product, when defined, is given by

$$uv = \sum_{i,j \geq -1} \Delta_i u \Delta_j v.$$

With this in mind, define the paraproduct, a symmetric bilinear operator, as follows:

$$T_u v = \sum_{j=-1}^{\infty} S_{j-2} u \Delta_j v,$$

and similarly, define the remainder as

$$R(u, v) = \sum_{|j-k| \leq 1} \Delta_j u \Delta_k v.$$

Therefore, when the product uv is defined, it can be written as

$$uv = T_u v + T_v u + R(u, v).$$

2.3 Useful Results and Lemmas

One of the most useful tools in working with Besov spaces is Bernstein's inequality, which I state below:

Proposition 2.3.1 (Bernstein's Inequality)**.** *Let* $f \in \mathcal{S}'$*. If supp* $\hat{f} \subset \{\xi : |\xi| \leq r\}$*, then there exists a constant* C *depending on the dimension* d*, such that for* $1 \leq p \leq q \leq \infty$*,*

$$\|f\|_q \leq C r^{n(\frac{1}{p} - \frac{1}{q})} \|f\|_p$$
$$\|D^\gamma f\|_p \leq C r^{|\gamma|} \|f\|_p .$$

If supp $\hat{f} \subset \{\xi : |\xi| \simeq r\}$*, then for* $1 \leq p \leq q \leq \infty$*,*

$$\|f\|_q \simeq r^{n(\frac{1}{p} - \frac{1}{q})} \|f\|_p$$
$$\|D^\gamma f\|_p \simeq r^{|\gamma|} \|f\|_p .$$

Proof. I follow the method of proof shown in [9]. Let ψ be a C_0^∞ function which is equal to 1 in the neighborhood of the ball centered at the origin with radius r. Let g be the function such that $\hat{g} = \psi$. By construction, $\hat{f}(\xi) = \psi(\lambda^{-1}\xi)\hat{f}(\xi)$, which gives that

$$f(x) = \lambda^d \int g(\lambda y) f(x-y)\mathrm{d}y := f * g_\lambda.$$

Furthermore, for every multi-index $\gamma, |\gamma| = k$ one has

$$D^\gamma f(x) = \lambda^{d+|\gamma|} \int (D^\gamma g)(\lambda y) f(x-y)\mathrm{d}y,$$

By Young's inequality, for $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$,

$$\|u * v\|_q \leq C\,\|u\|_r\,\|v\|_p\,,$$

which shows that $\|D^\gamma f\|_q \leq \lambda^{|\gamma|}\lambda^{d(1-\frac{1}{r})}\,\|D^\gamma g\|_r\,\|f\|_p$. To prove the first half of Bernstein's inequality, it remains only to control $\|D^\gamma g\|_r$. Using standard L^p-estimates, the definition of g, and the fact that the Fourier transform maps $\mathrm{L}^1 \longrightarrow \mathrm{L}^\infty$, one has

$$\begin{aligned}
\|D^\gamma g\|_r &\leq C\left(\|D^\gamma g\|_\infty + \|D^\gamma g\|_1\right)\\
&\leq C\left\|(1+|\cdot|^2)^d D^\gamma g\right\|_\infty\\
&\leq C\left\|(Id-\Delta)^d((\cdot)^\gamma\psi)\right\|_1\\
&\leq C^k.
\end{aligned}$$

The second half of Bernstein's inequality is proven similarly, but instead of a single bump function ψ centered at the origin, one has an infinite series of bump functions defined on S^{d-1}, whose support excludes the origin, and which form a partition of unity. The argument then follows a similar path to that described above, and is hence omitted. $\square$

Using Bernstein's inequality and the basic definition of Besov spaces, one has the following embeddings and equivalences (see [24] or [28] for more details):

Proposition 2.3.2. *The following embeddings hold:*

- $B^s_{p,q} \hookrightarrow B^{s_1}_{p,q_1}$ *if* $s_1 < s$ *or* $s_1 = s, q_1 \geq q$.
- *If* k *is an integer, then* $B^k_{p,1} \hookrightarrow W^{k,p} \hookrightarrow B^k_{p,\infty}$.
- $B^s_{p,q} \hookrightarrow B^{s_1}_{p_1,q}$ *if* $\frac{d}{p_1} - s_1 = \frac{d}{p} - s$, $p_1 \geq p$, $s_1 \leq s$.

Furthermore, the following equivalences between Besov spaces and other function spaces hold:

- *For the Sobolev space* H^s, *we have*

$$\|f\|_{H^s} \simeq \|\Delta_{-1} f\|_2 + \left(\sum_{j=0}^{\infty} 2^{2js} \|\Delta_j f\|_2^2\right)^{\frac{1}{2}} < \infty,$$

  i.e. $f \in H^s \Leftrightarrow f \in B^s_{2,2}$.

- *For the Hölder space* C^r, $0 < r < 1$,

$$\|f\|_{C^r} \simeq \sup_{j \geq -1} 2^j \|\Delta_j f\|_r$$

  .

It is a classical result (discussed in chapters 2 and 3 of [24]) that the dual of the Besov space $B^s_{p,q}$ is given by $B^{-s}_{p',q'}$, where

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Since B_Γ is defined based upon the $B^0_{\infty,1}$-norm, it makes sense that it's dual is defined with respect to the $B^0_{1,\infty}$-norm. In fact, the predual of B_Γ, which we denote H_Γ, is isomorphic to $(B_\Gamma)'$, and is defined as follows:

$$H_\Gamma = \left\{ f \in \mathcal{S}' \;\middle|\; \exists \, \{d_j\}_{j=-1}^\infty, \; d_j \geq 0, \; \sum_{j=-1}^{\infty} d_j < \infty \right.$$
$$\left. \text{and } \|\Delta_m f\|_1 \leq \sum_{j \geq m} d_j \Gamma(j)^{-1} \; \forall \, m \geq -1. \right\}$$

In order to apply the a priori estimate we find in the next chapter to the proofs of uniqueness and existence of $(B_{\kappa,0})$, one must first understand what happens when the Δ_j operator is applied to the nonlinear term $(u, \nabla)\rho$. I follow the general approach introduced in [1] and write

$$R_j(u, \rho) = \Delta_j (u, \nabla)\rho - (S_{j-2}u, \nabla)\Delta_j \rho.$$

Let M_0 be the constant such that $\Delta_j \Delta_k f = 0$ if $|j - k| > M_0$. (Note that M_0 depends strictly on our choice of φ and Φ defining the Δ_j operator.)

Theorem 2.3.3. *For u, ρ defined above:*

$$\|R_j(u,\rho)\|_\infty \leq C \sum_{|j-l| \leq M_0} \left\{ \|S_{l-2}\rho\|_\infty \|\Delta_l \nabla u\|_\infty + \|S_{l-2}\nabla u\|_\infty \|\Delta_l \rho\|_\infty \right\} \quad (2.1)$$
$$+ C 2^j \sideset{}{'}\sum_{\substack{l \geq j - M_0 \\ |l-m| \leq 1}} 2^{-l} \|\Delta_l \nabla u\|_\infty \|\Delta_m \rho\|_\infty$$

where the $\sum'$ *implies that for* $l=-1$*, the factor* $\|\Delta_l \nabla u\|_\infty$ *should be replaced by* $\|\Delta_{-1} u\|_\infty$.

Proof. The proof follows similarly to that of theorem 6.1 in [30]. I can decompose $R_j(u,\rho) = \sum_{q=1}^4 R_j^q(u,\rho)$ as follows:

$$\begin{aligned}
R_j^1(u,\rho) &= \sum_{k=1}^n \Delta_j T_{\partial_k \rho} u_k,\\
R_j^2(u,\rho) &= \sum_{k=1}^n [T_{u_k}\partial_k, \Delta_j]\rho,\\
R_j^3(u,\rho) &= \sum_{k=1}^n T_{u_k - S_{j-2}u_k} \partial_k \Delta_j \rho,\\
R_j^4(u,\rho) &= \sum_{k=1}^n \Delta_j R(u_k, \partial_k \rho) - R(S_{j-2}u_k, \Delta_j \partial_k \rho).
\end{aligned}$$

(For a more extensive development of this decomposition, see [1]) To prove the above estimate, I bound each term individually.

First, one can write $R_j^1(u,\rho)$ as

$$R_j^1(u,\rho) = \sum_{k=1}^n \sum_{|l-j|\le M_0} S_{l-2}\partial_k \rho \Delta_l u_k.$$

For $l<0$, $S_{l-2}\partial_k\rho = 0$ by definition of the S_{l-2} operator, so one can assume that $l \ge 0$ in the estimate of $R_j^1(u,\rho)$. Then using Bernstein's inequality, I have

$$\left\|R_j^1(u,\rho)\right\|_\infty \leq \sum_{k=1}^n \sum_{|l-j|\leq M_0} \|S_{l-2}\partial_k\rho\|_\infty \|\Delta_l u_k\|_\infty \tag{2.2}$$
$$\leq C \sum_{|l-j|\leq M_0} \|S_{l-2}\nabla\rho\|_\infty \|\Delta_l u\|_\infty$$
$$\leq C \sum_{|l-j|\leq M_0} 2^l \|S_{l-2}\rho\|_\infty 2^{-l} \|\Delta_l \nabla u\|_\infty$$
$$= C \sum_{|l-j|\leq M_0} \|S_{l-2}\rho\|_\infty \|\Delta_l \nabla u\|_\infty ,$$

where the third inequality follows from the fact that $l \geq 0$.

Writing out $R_j^2(u,\rho)$ explicitly, I use the fact that

div $S_{l-2}u = S_{l-2}$div $u = 0$ to compute

$$R_j^2(u,\rho) = \sum_{k=1}^n \sum_{|l-j|\leq M_0} [S_{l-2}u_k, \Delta_j]\partial_k\Delta_l\rho$$
$$= \sum_{k=1}^n \sum_{|l-j|\leq M_0} 2^{jn} \int \varphi(2^j(x-y))[S_{l-2}u_k(x) - S_{l-2}u_k(y)]\partial_k\Delta_l\rho(y)\mathrm{d}y$$
$$= \sum_{k=1}^n \sum_{|l-j|\leq M_0} 2^{jn} \int \partial_k\varphi(2^j(x-y))[S_{l-2}u_k(x) - S_{l-2}u_k(y)]\Delta_l\rho(y)\mathrm{d}y.$$

By Taylor's theorem, one has that the term in brackets is given by

$$[S_{l-2}u_k(x) - S_{l-2}u_k(y)] = \sum_{i=1}^n \int_0^1 S_{l-2}\partial_i u_k(x+\tau(y-x))\cdot(y_i-x_i)\mathrm{d}\tau,$$

which yields (using the change of variables $z \mapsto 2^j(x-y)$):

$$R_j^2(u,\rho) =$$
$$\sum_{k=1}^n \sum_{|l-j|\leq M_0} \sum_{i=1}^n \int \partial_k\varphi(z)z_i \int_0^1 S_{l-2}\partial_i u_k(x-\tau 2^{-j}z)\Delta_l\rho(x-2^{-j}z)\mathrm{d}\tau\mathrm{d}z.$$

Using Minkowski's inequality and the fact that $\varphi \in \mathcal{S}(\mathbb{R}^2)$, one has that $\left\|R_j^2(u,\rho)\right\|_\infty$ is bounded above by:

$$\begin{aligned}
&\sum_{k=1}^{n} \sum_{|l-j|\le M_0} \sum_{i=1}^{n} \int |\partial_k \varphi(z)|\,|z_i| \int_0^1 \left|S_{l-2}\partial_i u_k(x-\tau 2^{-j}z)\right| \left|\Delta_l \rho(x-2^{-j}z)\right| \mathrm{d}\tau \mathrm{d}z \\
&\qquad \le C \sum_{|l-j|\le M_0} \int |\nabla \varphi(z)|\,|z| \int_0^1 \left|S_{l-2}\nabla u(x-\tau 2^{-j}z)\right| \left|\Delta_l \rho(x-2^{-j}z)\right| \mathrm{d}\tau \mathrm{d}z \\
&\qquad \le C \sum_{|l-j|\le M_0} \|S_{l-2}\nabla u\|_\infty \|\Delta_l \rho\|_\infty . \qquad (2.3)
\end{aligned}$$

Next, for $R_j^3(u,\rho)$ one can write

$$\begin{aligned}
R_j^3(u,\rho) &= \sum_{k=1}^{n} T_{u_k - S_{j-2}u_k} \partial_k \Delta_j \rho \\
&= \sum_{k=1}^{n} \sum_{|l-j|\le M_0} S_{l-2}(u_k - S_{j-2}u_k)\partial_k \Delta_l \Delta_j \rho \\
&= \sum_{k=1}^{n} \sum_{|l-j|\le M_0} S_{l-2} \left(\sum_{m=j-1}^{j+M_0-2} \Delta_m u_k \right) \partial_k \Delta_l \Delta_j \rho.
\end{aligned}$$

For $l \ge 0$, observe that $\|\Delta_l u_k\|_\infty \le C 2^{-l} \|\Delta_l \nabla u\|_\infty$ while for $l=-1$, one has $\|\Delta_{-1} u_k\|_\infty \le C\|u\|_\infty$. This implies that

$$\begin{aligned}
\left\|R_j^3(u,\rho)\right\|_\infty &\le \begin{cases} C\sum_{|l-j|\le M_0} \|\Delta_l \nabla u\|_\infty \|\Delta_j \rho\|_\infty & \text{if } l \ge 0 \\ C\sum_{|j+1|\le M_0} \|u\|_\infty \|\Delta_j \rho\|_\infty & \text{if } l=-1, \end{cases} \\
&:= \sum_{|l-j|\le M_0}{}' \|\Delta_l \nabla u\|_\infty \|\Delta_j \rho\|_\infty . \qquad (2.4)
\end{aligned}$$

Next, I split $R_j^4(u,\rho)$ into two terms as follows:

$$
\begin{aligned}
R_j^4(u,\rho) &= \sum_{k=1}^n \Delta_j \partial_k R(u_k - S_{j-2}u_k, \rho) \\
&\qquad\qquad + \sum_{k=1}^n \{\Delta_j R(S_{j-2}u_k, \partial_k \rho) - R(S_{j-2}u_k, \Delta_j \partial_k \rho)\} \\
&: = R_j^{4,1}(u,\rho) + R_j^{4,2}(u,\rho).
\end{aligned}
$$

One can write the first term as

$$
R_j^{4,1}(u,\rho) = \sum_{k=1}^n \partial_k \Delta_j \sum_{|l-m|\le 1} \Delta_l (u_k - S_{j-2}u_k)\Delta_m \rho,
$$

which yields

$$
\begin{aligned}
\left\| R_j^{4,1}(u,\rho)\right\|_\infty &\le C \sum_{k=1}^n \left\| \Delta_j \sum_{|l-m|\le 1} \Delta_l (u_k - S_{j-2}u_k)\Delta_m \rho \right\|_\infty \\
&\le C \sum_{k=1}^n \sum_{l \ge j-M_0} 2^{j-l} \sum_{|l-m|\le 1}{}' 2^l \left\|\Delta_l u_k\right\|_\infty \left\|\Delta_m \rho\right\|)\infty \\
&\le C \sum_{l \ge j-M_0} 2^{j-l} \sum_{|l-m|\le 1}{}' \left\|\Delta_l \nabla u\right\|_\infty \left\|\Delta_m \rho\right\|_\infty .
\end{aligned} \tag{2.5}
$$

For the final term, I write $R_j^{4,2}(u,\rho)$ as:

$$
\begin{aligned}
R_j^{4,2}(u,\rho) &= \sum_{k=1}^n \sum_{|l-m|\le 1} \{\Delta_j (\Delta_l S_{j-2}u_k \Delta_j \Delta_m \partial_k \rho) - \Delta_l S_{j-2}u_k \Delta_m \partial_k \rho\} \\
&= \sum_{k=1}^n \sum_{|l-m|\le 1} \sum_{|l-j|\le M_0} [\Delta_j, \Delta_l S_{j-2}u_k]\Delta_m \partial_k \rho.
\end{aligned}
$$

Similar to the estimate of $R_j^2(u,\rho)$, I have:

$$
\begin{aligned}
\left\| R_j^{4,2}(u,\rho)\right\|_\infty &\le C \sum_{|l-j|\le M_0} \sum_{|l-m|\le 1} \left\|\Delta_l S_{j-2}\nabla u\right\| \infty \left\|\Delta_m \rho\right\|_\infty \\
&\le C \sum_{|l-j|\le M_0} \sum_{|l-m|\le 1} \left\|\Delta_l \nabla u\right\|_\infty \left\|\Delta_m \rho\right\|_\infty .
\end{aligned} \tag{2.6}
$$

Combining (2.2)-(2.6), the desired bound is achieved and the theorem is proved. $\square$

Next, I use the Biot-Savart law and properties of Γ to establish the following estimate on the $B^0_{\infty,1}$ of velocity:

Lemma 2.3.4. *Let $\omega \in L^{p_0} \cap B_\Gamma$ for $1 < p_0 < d$, and define $u = \mathcal{K} * \omega$ where $\mathcal{K}$ is the Biot-Savart kernel. Then the following bound holds:*

$$\|u\|_{B^0_{\infty,1}} \le C(\|\omega\|_{p_0} + \|\omega\|_\Gamma). \tag{2.7}$$

Proof. Using Littlewood-Paley decomposition and Bernstein's inequality, one can write

$$\begin{aligned}\|u\|_{B^0_{\infty,1}} &\le \|\Delta_{-1}u\|_\infty + \sum_{j=0}^{\infty} \|\Delta_j u\|_\infty \\ &\le \|\Delta_{-1}u\|_\infty + C\sum_{j=0}^{\infty} 2^{-j} \|\Delta_j \omega\|_\infty .\end{aligned}$$

For the first term, let $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\chi \equiv 1$ near the origin. Choose q_0 such that $\frac{1}{q_0} + \frac{1}{p_0} = 1$. Then one can write

$$\begin{aligned}\|\Delta_{-1}u\|_\infty &\le \|(\chi\mathcal{K}) * \Delta_{-1}\omega\|_\infty + \|((1-\chi)\mathcal{K}) * \Delta_{-1}\omega\|_\infty \\ &\le \|\chi\mathcal{K}\|_1 \|\Delta_{-1}\omega\|_\infty + \|(1-\chi)\mathcal{K}\|_{q_0} \|\Delta_{-1}\omega\|_{p_0} \\ &\le C\|\Delta_{-1}\omega\|_{p_0},\end{aligned} \tag{2.8}$$

since $\chi\mathcal{K} \in \mathrm{L}^1$ and $(1-\chi)\mathcal{K} \in \mathrm{L}^{q_0}$. For the second term, set

$$d_m = \sum_{j=1}^{m} \|\Delta_j \omega\|_\infty ,$$

and use Abel's lemma (i.e., summation by parts) to write

$$\begin{aligned}\sum_{j=0}^{\infty} 2^{-j} \left\|\Delta_j \omega\right\|_\infty &\leq \sum_{j=0}^{\infty} 2^{-j}(d_j - d_{j-1}) \\ &\leq -2^{-1}d_0 + \sum_{j=0}^{\infty} d_j(2^{-j} - 2^{-(j+1)}) \\ &\leq -2^{-1}d_0 + \|\omega\|_\Gamma \sum_{j=0}^{\infty} \Gamma(j)2^{-j} \\ &\leq C\Gamma(0) \|\omega\|_\Gamma = \|\omega\|_\Gamma ,\end{aligned} \tag{2.9}$$

where the last line follows from properties (i)-(iii) of $\Gamma(\alpha)$. Combining (2.8) and (2.9) yields the desired estimate on $\|u\|_{B^0_{\infty,1}}$. $\square$

Next, I introduce a key lemma of Chemin, which I will use to bound the growth of density in the next chapter:

Proposition 2.3.5 (Chemin)**.** *Let $j \geq 0$. Then there exists positive constants (C, c) such that for any $1 \leq p \leq \infty$ and $\nu > 0$,*

$$\left\|e^{\nu\Delta}\Delta_j h\right\|_p \leq Ce^{-c\nu 2^{2j}} \left\|\Delta_j h\right\|_p .$$

Proof. Set $\Delta_j h = u$. Let $\psi \in \mathcal{D}(\mathbb{R}^n \backslash \{0\})$ be identically equal to 1 on the support of $\hat{\varphi}$, the function used to define the Littlewood-Paley operator. (Explicitly, ψ is supported on $\{\xi | \frac{1}{2} \leq |\xi| \leq 2\}$.) Then one has:

$$\begin{aligned}e^{t\Delta}u &= \psi(\nu^{-1}D)e^{t\Delta}u \\ &= \mathcal{F}^{-1}\left(\psi(\nu^{-1}\xi)e^{-t|\xi|^2}\hat{u}(\xi)\right) \\ &= \tilde{f}_\nu(t,\cdot) * u,\end{aligned}$$

where $\tilde{f}_\nu(t,x) = \int e^{i(x\cdot\xi)}\psi(\nu^{-1}\xi)e^{-t|\xi|^2}\mathrm{d}\xi$. If I can show the existence of strictly positive constants (C,c) such that

$$\left\|\tilde{f}_\nu(t,\cdot)\right\|_1 \leq Ce^{-ct\nu^2}, \tag{2.10}$$

then by Young's inequality the proposition will be proven. To prove (2.10), define

$$\tilde{f}_\nu(t,x) = \nu^n f_\nu(t,\nu x),$$

where $f_\nu(t,x) = \int e^{i(x\cdot\xi)}\psi(\xi)e^{-t\nu^2|\xi|^2}\mathrm{d}\xi$. Using integration by parts, one has

$$\begin{aligned}
f_\nu(t,x) &= (1+|x|^2)^{-n}\int (1+|x|^2)^n e^{i(x\cdot\xi)}\psi(\xi)e^{-t\nu^2|\xi|^2}\mathrm{d}\xi \\
&= (1+|x|^2)^{-n}\int \left[(\mathrm{Id}-\Delta_\xi)^n\left(e^{i(x\cdot\xi)}\right)\right]\psi(\xi)e^{-t\nu^2|\xi|^2}\mathrm{d}\xi \\
&= (1+|x|^2)^{-n}\int e^{i(x\cdot\xi)}\left[(\mathrm{Id}-\Delta_\xi)^n\left(\psi(\xi)e^{-t\nu^2|\xi|^2}\right)\right]\mathrm{d}\xi.
\end{aligned}$$

By Leibnitz's formula,

$$(\mathrm{Id}-\Delta_\xi)^n\left(\psi(\xi)e^{-t\nu^2|\xi|^2}\right) = \sum_{\substack{|\alpha|\leq 2n \\ \beta\leq\alpha}} C_{\alpha,\beta}\partial^{(\alpha-\beta)}\psi(\xi)\partial^\beta(e^{-t\nu^2|\xi|^2}).$$

Furthermore, Faà di Bruno's formula gives

$$e^{t\nu|\xi|^2}\partial^\beta(e^{-t\nu^2|\xi|^2}) = \sum_{\substack{\gamma_1+\ldots+\gamma_m=\beta \\ |\gamma_j|\geq 1}} (-t\nu)^m\Pi_{j=1}^m\partial^{\gamma_j}(|\xi|^2).$$

Since the support of ψ is contained in an annulus, there exists some strictly positive constants, (C,c), such that for all ξ in supp ψ,

$$\begin{aligned}
\left|\partial^\beta(e^{-t\nu^2|xi|^2})\right| &\leq C^{|\beta|}(1+t\nu^2)^{|\beta|}e^{-t\nu^2|\xi|^2} \\
&\leq C^{|\beta|}(1+t\nu^2)^{|\beta|}e^{-ct\nu^2}.
\end{aligned}$$

This yields $f_\nu(t,x) \le C(1+|x|^2)^{-n} e^{-ct\nu^2}$, which combined with the definition of $\tilde{f}_\nu(t,x)$ gives the desired L^1-bound. $\square$

Finally, I will use approximation by Sobolev-regular solutions in the proof of existence for the $(B_{\kappa,0})$, for which I need the following result from [6]:

Proposition 2.3.6 (Chae)**.** *Let $\kappa > 0$ be fixed, and div $u_0 = 0$. Let $r > 2$ be an integer, and $(u_0, \rho_0) \in H^r(\mathbb{R}^2)$. Then there exists a unique solution (u, ρ) to $(B_{\kappa,0})$ with $u \in C([0,\infty); H^r(\mathbb{R}^2))$ and $\rho \in C([0,\infty); H^r(\mathbb{R}^2)) \cap \mathrm{L}^2([0,T]; H^{r+1}(\mathbb{R}^2))$.*

Chapter 3

Main Result

In this chapter, I prove the existence and uniqueness of solutions to the two-dimensional Boussinesq equations under the assumption that the initial data belongs to B_Γ. Explicitly, I prove the following theorems:

Theorem 3.0.7. *For $1 < p_0 < 2 < p_1 < \infty$, let $f \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ and $g \in \mathrm{W}^{1,p_0} \cap \mathrm{W}^{1,p_1}$ such that $\nabla g \in B_\Gamma$. Assume that*

$$(\alpha + 2)\Gamma'(\alpha) \leq C \text{ for a.e. } \alpha \in [-1, \infty).$$

Then there exists a $T > 0$ (depending on Γ, f and g) and a unique solution (u, ρ) to the system of equations $(B_{\kappa,0})$ with $u = \mathcal{K} * \omega$, such that

$$\omega(\,\cdot\,) \in \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0,T]; B_{\Gamma_1}), \tag{3.1}$$

$$\nabla\rho(\,\cdot\,) \in \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0,T]; B_\Gamma). \tag{3.2}$$

Theorem 3.0.8. *Let f and g be as in theorem 3.3.1. Assume that*

$$\Gamma'(\alpha)\Gamma_1(\alpha) \leq C \text{ for a.e. } \alpha \in [-1, \infty).$$

Then there exist a unique (u, ρ) solving $(B_{\kappa,0})$ such that

$$\omega(\,\cdot\,) \in \mathrm{L}^\infty_{loc}([0,\infty); \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0,\infty); B_{\Gamma_1}), \tag{3.3}$$

$$\nabla\rho(\,\cdot\,) \in \mathrm{L}^\infty_{loc}([0,\infty); \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0,\infty); B_\Gamma). \tag{3.4}$$

As mentioned in the introduction, the structure of the proof is as follows - I show that for positive time, the growth of vorticity and density can be bounded a priori by a locally integrable function. Using this, I next prove uniqueness by showing that for two solutions, the $B^{\infty}_{0,1}$ norm of their difference is bounded above by a monotonically increasing, nonnegative, absolutely continuous function. Using some results from ODE theory, I prove by contradiction that that function is identically zero on a positive time interval, hence uniqueness must hold at least locally in time and, under the proper assumptions, globally in time as well. Note that the uniqueness result holds for a weaker choice of function $\Gamma(\alpha)$ than that described in section 2.2. Finally, using the existence of solutions in Sobolev spaces shown by Chae in [6], I prove the existence of solutions to $(B_{\kappa,0})$ given initial data in $B_\Gamma(\mathbb{R}^2) \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ using an extension of the argument used to show uniqueness, along with a utilization of the dual-space of B_Γ to show that said solutions are weak-* continuous in time with values in B_Γ, B_{Γ_1}, respectively.

To begin, I study the structure of the vorticity equation derived from the Boussinesq equations and establish some a priori estimates on the growth of vorticity and density.

3.1 A Priori Estimates

The goal of this section is to prove the following theorem:

Theorem 3.1.1. *Let* $1 < p_0 < 2 < p_1 < \infty$. *Let* $f \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$, *and let* $g \in \mathrm{W}^{1,p_0} \cap \mathrm{W}^{1,p_1}$ *such that* $\nabla g \in B_\Gamma$. *Let* $u = \mathcal{K} * \omega$, $\omega_0 = f$ *and* $\rho_0 = g$.

Finally, let (u, ρ) solve $(B_{\kappa,0})$. then we have the following a priori estimates:

1. *There exists $T > 0$ (depending on Γ, f and g) such that $\omega, \nabla\rho \in \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1})$, $\nabla\rho \in \mathrm{L}^\infty([0,T]; B_\Gamma)$, and $\omega \in \mathrm{L}^\infty([0,T]; B_{\Gamma_1})$ when*

$$(\alpha + 2)\Gamma'(\alpha) \leq C \text{ for a.e. } \alpha \in [-1, \infty). \tag{3.5}$$

2. *Furthermore, $\omega, \nabla\rho \in \mathrm{L}^\infty_{loc}([0,\infty); \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1})$, $\nabla\rho \in \mathrm{L}^\infty_{loc}([0,\infty); B_\Gamma)$, and $\omega \in \mathrm{L}^\infty_{loc}([0,\infty); B_{\Gamma_1})$ under the stronger assumption that*

$$\Gamma'(\alpha)\Gamma_1(\alpha) \leq C \text{ for a.e. } \alpha \in [-1, \infty). \tag{3.6}$$

Observe that if I apply the curl operator to the equation satisfied by the velocity field u then curl $\nabla P = 0$, and I have the vorticity equation

$$\partial_t\omega + (u, \nabla)\omega = \partial_1\rho.$$

Integrating along characteristic curves, one has that for flow map $X(x,t;\tau) = X_u(x,t;\tau)$,

$$\omega(x(t),t) = \omega_0(X^{-1}(t;0)x(t)) + \int_0^t \partial_1\rho(X^{-1}(t;\tau)x(t),\tau)\mathrm{d}\tau.$$

For any $p \in [1,\infty)$, this implies

$$\|\omega(t)\|_p \leq \|\omega_0\|_p + \int_0^t \|\nabla\rho\|_p \,\mathrm{d}\tau, \tag{3.7}$$

since X is a volume-preserving homeomorphism. In regards to the B_{Γ_1} norm, an initial estimate gives:

$$\|\omega(t)\|_{\Gamma_1} \leq \left\|\omega_0(X^{-1}(t))\right\|_{\Gamma_1} + \int_0^t \left\|\nabla\rho(X^{-1}(t;\tau),\tau)\right\|_{\Gamma_1} \mathrm{d}\tau. \tag{3.8}$$

With (3.8) in mind, I must address the following two concerns:

1. Does the gradient of density remain in the initial B_Γ space for positive time?

2. Assuming this is the case, how does the inverse flow map X^{-1} act on the B_Γ spaces?

One of the main results in Vishik's paper [31] is the following proposition, in which he addresses the second question in the setting of the Euler equations. He demonstrates that:

Proposition 3.1.2. *Let $\omega_0 \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$, and let $\rho_0 \in \mathrm{W}^{1,p_0} \cap \mathrm{W}^{1,p_1}$ such that $\nabla\rho_0 \in B_\Gamma$. Let (u,ρ) be a regular solution to $(B_{\kappa,0})$ such that $u \in \mathcal{K} * C([0,T]; B_{\Gamma_1} \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1})$, where T is defined for each case below. Let $X_u(t;\tau)$ be the flow map given by u, and let $f \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ be an arbitrary function. Then one has the following estimates:*

1. *If (3.5) holds, then there exists a $T > 0$ and a $C > 0$ (both depending on Γ and the initial data), such that for $0 \le \tau \le t \le T$,*

$$\left\| f \circ X_u^{-1}(t;\tau) \right\|_{\Gamma_1} \le C \left\| f \right\|_\Gamma .$$

2. *Under assumption (3.6), let $T > 0$ be arbitrary. Then there exists $\lambda(\cdot) \in \mathrm{L}^\infty_{loc}(0,\infty)$ (depending on Γ and the initial data), such that:*

$$\left\| f \circ X_u^{-1}(t;\tau) \right\|_{\Gamma_1} \le C \left\| f \right\|_\Gamma \lambda(t)$$

for all $0 \le \tau \le t \le T$.

Note that discussion of the proofs involved in Vishik's result are contained in the appendix to this thesis. In his paper, Vishik proves the above result under the more specific assumptions of the 2D incompressible Euler system, wherein one is evolving the initial vorticity - the only quantity that needs controlling is

$$\omega(t) = \omega_0(X^{-1}(t;\tau), t).$$

With respect to Vishik's original result, however, one can make two observations: first, that his result (stated in it's original form in proposition A.0.5), may be extended to the Boussinesq system, because with respect to the involved quantities the estimate primarily utilizes the Biot-Savart relationship between velocity and vorticity. Second, the flow can be completely decoupled from the function it is evolving - one needs only that that evolving function belong to the proper function spaces, not that it be related to the velocity field as is the case of vorticity. With this in mind, Vishik's proposition, shown with respect to the Euler equations in [31], can be made relevant to the Boussinesq equations in the version stated above.

Remark 3.1.1. For ease of exposition, I fix $T > 0$ for the remainder of this chapter. In the case of assumption (3.5), this T depends on the choice of Γ, while under assumption (3.6), this choice of T is arbitrary. For more discussion of the distinction between these two cases, see the appendix.

The second tool I need to prove theorem 3.1.1 is the following a priori control on the density ρ, which answers the first question regarding the membership of the density:

Theorem 3.1.3. *Assume ω_0 and ρ_0 are defined as above. Set $\alpha = \left(\frac{1+\kappa t}{\kappa}\right)$. Then there is a constant $C > 0$ (depending only on Γ, f, g and T) such that for all $t > 0$, $\int_0^t \|\nabla\rho(\tau)\|_{\mathrm{L}^{p_0}\cap B_\Gamma} d\tau \leq \Upsilon(t)$, where*

$$\Upsilon(t) = \left[\alpha^2 \left(\|\rho_0\|^2_{B^{-1}_{p_0,1}\cap B^{-1}_{\infty,1}} + \alpha t \|\rho_0\|_2^2 \|\omega_0\|^2_{\mathrm{L}^{p_0}\cap B_\Gamma}\right)\right]^{\frac{1}{2}} \exp\left[C\alpha^3 \|\rho_0\|_2^2 t\right].$$

Using proposition 3.1.2 together with theorem 3.1.3, one can conclude that for $t \in [0, T]$, the terms on the right hand side of (3.8) are bounded by a constant multiple of $\|\omega_0\|_\Gamma$ and $\int_0^t \|\nabla\rho(\tau)\|_\Gamma \mathrm{d}\tau$, respectively, hence proving theorem 3.1.1.

It remains to prove theorem 3.1.3. First, note that ρ solves

$$\partial_t\rho - \kappa\Delta\rho = h,$$

where $h = -(u, \nabla)\rho$. Written in this form, it becomes evident that one can treat the right hand side as the non-homogeneous part of a heat equation and make use of the smoothing properties of the Laplacian. Let $\{e^{\nu\Delta}\}_{\nu>0}$ stand for the heat semi-group. Applying the Δ_j operator to the above equation, one has

$$\partial_t\Delta_j\rho - \kappa\Delta\Delta_j\rho = \Delta_j h, \tag{3.9}$$

which implies that

$$\Delta_j\rho(t) = e^{\kappa t\Delta}\Delta_j\rho_0 + \int_0^t e^{\kappa(t-\tau)\Delta}\Delta_j h(\tau)\mathrm{d}\tau. \tag{3.10}$$

For the case $j = -1$, I apply the maximum principle to (3.9) and find

$$\begin{aligned} \|\Delta_{-1}\rho(t)\|_p &\le \|\Delta_{-1}\rho_0\|_p + \int_0^t \|\Delta_{-1}h(\tau)\|_p \,\mathrm{d}\tau \\ \Rightarrow \int_0^t \|\Delta_{-1}\rho(\tau)\|_p \,\mathrm{d}\tau &\le Ct\left(\|\Delta_{-1}\rho_0\|_p + \int_0^t \|\Delta_{-1}h(\tau)\|_p \,\mathrm{d}\tau\right). \end{aligned} \tag{3.11}$$

For $j \ge 0$, proposition 2.3.5 applied to (3.10) gives me

$$\|\Delta_j\rho(t)\|_p \le C\left(e^{-C\kappa 2^{2j}t}\|\Delta_j\rho_0\|_p + \int_0^t e^{-C\kappa 2^{2j}(t-\tau)}\|\Delta_j h(\tau)\|_p \,\mathrm{d}\tau\right), \tag{3.12}$$

and integrating both sides in t leads to the following inequality for $j \ge 0$:

$$\kappa 2^j \int_0^t \|\Delta_j\rho(\tau)\|_p \,\mathrm{d}\tau \le C2^{-j}\left(\|\Delta_j\rho_0\|_p + \int_0^t \|\Delta_j h(\tau)\|_p \,\mathrm{d}\tau\right). \tag{3.13}$$

Let $1 \le p \le \infty$. Combining (3.13) and (3.11) and summing $j = -1$ to ∞, I conclude that for any $1 \le p \le \infty$,

$$\kappa \int_0^t \|\rho(\tau)\|_{B^1_{p,1}} \,\mathrm{d}\tau \le C(1+\kappa t)\left(\|\rho_0\|_{B^{-1}_{p,1}} + \int_0^t \|(u,\nabla)\rho\|_{B^{-1}_{p,1}} \,\mathrm{d}\tau\right). \tag{3.14}$$

Observe that for $p = \infty$, proposition 2.3.2 and Bernstein's inequality imply that the norm on the left hand side of (3.14) is equivalent to the time integral of the $B^0_{\infty,1}$-norm of $\nabla\rho$, which is itself an upper bound for time integral of $\|\nabla\rho\|_\Gamma$. Similarly, for $p = p_0$, the $B^1_{p_0,1}$-norm of ρ on the left hand side is a bound for $\|\nabla\rho\|_{p_0}$. Therefore, in order to prove the a priori bound on the $B_\Gamma \cap \mathrm{L}^{p_0}$-norm of $\nabla\rho$, it suffices to control the right hand side of (3.14) by suitable bounds and then utilize a Gronwall-type estimate.

Remark 3.1.2. For the space $B^{-1}_{p_0,1}$, I use the embedding $\mathrm{W}^{1,p_0} \hookrightarrow B^1_{p_0,\infty} \hookrightarrow B^{-1}_{p_0,1}$ given by proposition 2.3.2.

Since the cases $p = \infty$ and $p = p_0$ are nearly identical, I address only the case $p = \infty$. Use Bony's paraproduct decomposition to write

$$(u, \nabla)\rho = R(u, \nabla\rho) + \sum_{m=1}^{2} T_{\partial_m \rho} u_m + T_{u_m} \partial_m \rho,$$

where $R(f, g) = \sum_{|j-k|\leq 1} \Delta_j f \Delta_k g$, and $T_f g = \sum_{j=0}^{\infty} S_{j-2} f \Delta_j g$. Since div $u = 0$, I have $R(u, \nabla\rho) =$ div $R(u, \rho)$. Because the div operator maps $B^0_{\infty,1} \longrightarrow B^{-1}_{\infty,1}$, it suffices to bound $\|R(u, \rho)\|_{B^0_{\infty,1}}$. To do so, I write

$$\begin{aligned}
\|R(u, \rho)\|_{B^0_{\infty,1}} &= \sum_{j=-1}^{\infty} \left\| \Delta_j \sum_{|k-l|\leq 1} \Delta_k u \Delta_l \rho \right\|_{\infty} \\
&\leq \sum_{j=-1}^{\infty} \sum_{\substack{|k-l|\leq 1 \\ |j-k|\leq M_0}} \|\Delta_j \Delta_k u \Delta_l \rho\|_{\infty} \\
&\leq C \sum_{j=-1}^{\infty} \sum_{|j-l|\leq M_0} \|\Delta_j u\|_{\infty} \|\Delta_l \rho\|_{\infty} \\
&\leq C \|\rho\|_{B^0_{\infty,\infty}} \|u\|_{B^0_{\infty,1}},
\end{aligned}$$

where $\|\rho\|_{B^0_{\infty,\infty}} = \sup_{j\geq -1} \|\Delta_j \rho\|_{\infty}$. By lemma 2.3.4, I conclude that

$$\|R(u, \nabla\rho)\|_{B^{-1}_{\infty,1}} \leq C \|\rho\|_{B^0_{\infty,\infty}} (\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1}). \tag{3.15}$$

For the second term in the paraproduct decomposition, I use the following estimate:

$$\begin{aligned}
\|T_{\partial_m \rho} u_m\|_{B^{-1}_{\infty,1}} &= \sum_{j=-1}^{\infty} 2^{-j} \left\| \Delta_j \left(\sum_{k=0}^{\infty} S_{k-2} \partial_m \rho \Delta_k u_m\right) \right\|_{\infty} \\
&\leq C \sum_{j=-1}^{\infty} \sum_{|j-k|\leq M_0} 2^{-j} \|\Delta_j \partial_m \rho\|_{\infty} \|\Delta_k u_m\|_{\infty}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\,\|\rho\|_{B^0_{\infty,\infty}} \sum_{k=-1}^{\infty} \|\Delta_k u\|_\infty \\
&\leq C\,\|\rho\|_{B^0_{\infty,\infty}} \|u\|_{B^0_{\infty,1}}\,.
\end{aligned}
$$

The above is true regardless of my choice of m. Similarly for the $B^{-1}_{\infty,1}$-norm of $T_{u_m}\partial_m\rho$:

$$
\begin{aligned}
\|T_{u_m}\partial_m\rho\|_{B^{-1}_{\infty,1}} &= \sum_{j=-1}^{\infty} 2^{-j} \left\| \Delta_j\Big(\sum_{k=0}^{\infty} S_{k-2}u_m \Delta_k \partial_m \rho\Big) \right\|_\infty \\
&\leq C \sum_{j=-1}^{\infty} \sum_{|j-k|\leq M_0} \|\Delta_j u_m\|_\infty 2^{-k} \|\Delta_k \partial_m \rho\|_\infty \\
&\leq C\,\|\rho\|_{B^0_{\infty,\infty}} \sum_{j=-1}^{\infty} \|\Delta_j u\|_\infty \\
&\leq C\,\|\rho\|_{B^0_{\infty,\infty}} \|u\|_{B^0_{\infty,1}}\,.
\end{aligned}
$$

Combining these three estimates and summing over m, I find

$$
\left\| \sum_{m=1}^{2} T_{\partial_m\rho} u_m + T_{u_m}\partial_m\rho \right\|_{B^{-1}_{\infty,1}} \leq C\,\|\rho\|_{B^0_{\infty,\infty}} (\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1}). \tag{3.16}
$$

By Hölder's inequality, one can write

$$
\int_0^t \|\rho\|_{B^0_{\infty,\infty}} (\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1}) \mathrm{d}\tau \leq \tag{3.17}
$$

$$
\left(\int_0^t \|\rho\|^2_{B^0_{\infty,\infty}} \mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_0^t (\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1})^2 \mathrm{d}\tau\right)^{\frac{1}{2}},
$$

and bound each integral individually. To handle the first integral, observe that if I take the L^2-inner product of ρ with the equation satisfied by ρ, I have

$$
\langle \rho, \partial_t \rho\rangle + \langle \rho, (u,\nabla)\rho\rangle + \langle \rho, \kappa\Delta\rho\rangle = 0.
$$

Following an integration by parts in the space variable and a time integration over [0,t], one has

$$\|\rho(t)\|_2^2 + 2\kappa \int_0^t \|\nabla\rho(\tau)\|_2^2 \,\mathrm{d}\tau = \|\rho_0\|_2^2 \tag{3.18}$$

for all $t \in \mathbb{R}_+$. By the definition of $\|\rho\|_{B^0_{\infty,\infty}}$ and Bernstein's inequality,

$$\begin{aligned} \|\rho\|_{B^0_{\infty,\infty}} &= \sup_{j\geq -1} \|\Delta_j \rho\|_\infty \\ &\leq \sup_{j\geq -1} 2^j \|\Delta_j \rho\|_2 \\ &\leq C\|\Delta_{-1}\rho\|_2 + \sup_{j\geq 0} 2^j \|\Delta_j \rho\|_2 \\ &\leq C(\|\rho\|_2 + \|\nabla\rho\|_2). \end{aligned} \tag{3.19}$$

To utilize (3.18), I square both sides of (3.19) and integrate in time, to conclude

$$\begin{aligned} \int_0^t \|\rho\|^2_{B^0_{\infty,\infty}} \,\mathrm{d}\tau &\leq C\int_0^t \|\rho\|_2^2 \,\mathrm{d}\tau + \int_0^t \|\nabla\rho\|_2^2 \,\mathrm{d}\tau \\ &\leq C\|\rho_0\|_2^2\, t + \frac{1}{\kappa}\|\rho_0\|_2^2 \\ &\leq C\alpha \|\rho_0\|_2^2 . \end{aligned}$$

Combining the above with the bound given by (3.14), I find that

$$\begin{aligned} &\int_0^t \|\nabla\rho(\tau)\|_\Gamma \,\mathrm{d}\tau \\ &\quad\leq C\alpha\left[\|\rho_0\|_{B^{-1}_{\infty,1}} + C\int_0^t \|\rho\|_{B^0_{\infty,\infty}} (\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1})\mathrm{d}\tau\right] \\ &\quad\leq C\alpha\left[\|\rho_0\|_{B^{-1}_{\infty,1}} + C\alpha^{\frac{1}{2}}\|\rho_0\|_2 \left(\int_0^t (\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1})^2 \mathrm{d}\tau\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{3.20}$$

To achieve the desired a priori bound on the gradient of the density, it then suffices to control the two norms on the right hand side of (3.20) that are not bounds on initial data - the L^2 (in time) integral of the L^{p_0} and B_{Γ_1} norms (in space) of vorticity. By (3.7) and proposition 3.1.2, one has for all $t \in [0, T]$:

$$\|\omega(t)\|_{p_0} \leq \|\omega_0\|_{p_0} + \int_0^t \|\nabla\rho\|_{p_0} \mathrm{d}\tau, \tag{3.21}$$

$$\|\omega(t)\|_{\Gamma_1} \leq C\left(\|\omega_0\|_{\Gamma} + \int_0^t \|\nabla\rho(\tau)\|_{\Gamma} \mathrm{d}\tau\right). \tag{3.22}$$

Define $\Theta(t) = \int_0^t \|\nabla\rho(\tau)\|_{B_\Gamma\cap\mathrm{L}^{p_0}} \mathrm{d}\tau$, where $\|\cdot\|_{B_\Gamma\cap\mathrm{L}^{p_0}} = \max\left\{\|\cdot\|_{p_0}, \|\cdot\|_{\Gamma}\right\}$. Then inserting the above into (3.20) gives

$$\int_0^t \|\nabla\rho(\tau)\|_{\Gamma} \mathrm{d}\tau \leq C\alpha\left[\|\rho_0\|_{B^{-1}_{\infty,1}} + \right.$$
$$\left. + C\,\alpha^{\frac{1}{2}}\|\rho_0\|_2\left(t\|\omega_0\|^2_{\mathrm{L}^{p_0}\cap B_\Gamma} + C\int_0^t \Theta^2(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\right]. \tag{3.23}$$

The argument with respect to $B^0_{p_0,1}$ yields an identical estimate, with $\|\nabla\rho\|_{p_0}$ and $\|\rho_0\|_{B^{-1}_{p_0,1}}$ replacing the first two norms in (3.23), respectively, and therefore one has:

$$\int_0^t \|\nabla\rho\|_{p_0} \mathrm{d}\tau \leq C\alpha\left[\|\rho_0\|_{B^{-1}_{p_0,1}} + \right.$$
$$\left. + C\,\alpha^{\frac{1}{2}}\|\rho_0\|_2\left(t\|\omega_0\|^2_{\mathrm{L}^{p_0}\cap B_\Gamma} + C\int_0^t \Theta^2(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\right]. \tag{3.24}$$

Next, I combine (3.23), (3.24) and square both sides, to find

$$\begin{aligned}\Theta^2(t) &\leq C\alpha^2\left[\|\rho_0\|^2_{B^{-1}_{p_0,1}\cap B^{-1}_{\infty,1}} + C\alpha\|\rho_0\|_2^2\left(t\|\omega_0\|^2_{\mathrm{L}^{p_0}\cap B_\Gamma} + C\int_0^t \Theta^2(\tau)\mathrm{d}\tau\right)\right] \\ &\leq C\alpha^2\left[\|\rho_0\|^2_{B^{-1}_{p_0,1}\cap B^{-1}_{\infty,1}} + \alpha t\|\rho_0\|_2^2\|\omega_0\|^2_{\mathrm{L}^{p_0}\cap B_\Gamma}\right] \\ &\qquad + C\alpha^3\|\rho_0\|_2^2\int_0^t \Theta^2(\tau)\mathrm{d}\tau.\end{aligned}$$

An application of Gronwall's inequality to $\Theta^2(t)$ gives

$$\Theta^2(t) \le C\alpha^2 \left(\|\rho_0\|^2_{B^{-1}_{p_0,1} \cap B^{-1}_{\infty,1}} + \alpha t \|\rho_0\|_2^2 \|\omega_0\|^2_{\mathrm{L}^{p_0} \cap B_\Gamma} \right) \exp \left[C\alpha^3 \|\rho_0\|_2^2 t \right]$$

and taking a square root of both sides yields the desired bound, thus completing the proof of theorem 3.1.3.

3.2 Uniqueness of the flow

Let $\Pi : \mathbb{R} \to [1, \infty)$ be a function such that (i)-(iii) of section 2.2 are satisfied. In addition, assume the following holds for Π:

$$\int_1^\infty [\Pi(\xi)]^{-1} \mathrm{d}\xi = \infty, \tag{3.25}$$

$$\Pi(\xi) 2^{-\xi} \text{ is non-increasing for } \xi \ge C \text{ , } \lim_{\xi \to \infty} \Pi(\xi) 2^{-\xi} = 0. \tag{3.26}$$

Remark 3.2.1. I use Π in place of Γ and Γ_1 in this section since the uniqueness result utilizes weaker assumptions on Π than those needed in section 3.3. I do not require that $(\xi + 2)\Pi(\xi)$ be convex, only that the tail of $\Pi(\xi)$ grow slower than 2^ξ at infinity.

Theorem 3.2.1. *Let $(u_1, \rho_1), (u_2, \rho_2)$ be two solutions to $(B_{\kappa,0})$, and let $\omega_{1,2} =$ curl $u_{1,2}$. Assume that for $1 < p_0 < 2$:*

$$\omega_{1,2}, \nabla\rho_{1,2} \in \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0}), \ \ \|\omega_{1,2}(\cdot)\|_\Pi , \|\nabla\rho_{1,2}(\cdot)\|_\Pi \in \mathrm{L}^\infty([0,T]), \tag{3.27}$$

$$u_{1,2} = \mathcal{K} * \omega_{1,2}, \tag{3.28}$$

$$\mathit{div}\ u_{1,2} = 0 \tag{3.29}$$

$$\omega_{1,2}(\cdot, 0) = f(\cdot), \rho_{1,2}(\cdot, 0) = g(\cdot);\ f, g \in B_\Pi \cap \mathrm{L}^{p_0}. \tag{3.30}$$

Then $(u_1, \rho_1) = (u_2, \rho_2)$ for $t \in [0, T]$.

The proof of uniqueness relies on a close study of the effects of the Δ_j operator on the nonlinear term $(u, \Delta_j)f$ for f either density and velocity. Using paradifferential calculus and Littlewood-Paley theory, one is able to control the growth of the sum of the B_Π-norms of density and velocity by a function of time which is absolutely continuous, monotone and nondecreasing. I then follow an argument similar to that used by Vishik in [31] and demonstrate that there exists a nonzero interval of time such that this function must be identically zero on that interval - hence uniqueness must hold on that interval.

Proof. Define $v = u_1 - u_2$, $\omega = \omega_1 - \omega_2$, $\rho = \rho_1 - \rho_2$ and $P = P_1 - P_2$. One then has (for $\dot{f} = \frac{\partial}{\partial t} f$):

$$\begin{cases} \dot{v} = -(u_1, \nabla)v - (v, \nabla)u_2 - \nabla P + \begin{pmatrix} 0 \\ \rho \end{pmatrix} \\ \dot{\rho} - \kappa\Delta\rho = -(u_1, \nabla)\rho + (v, \nabla)\rho_2 \\ \text{div } v = 0 \\ v|_{t=0} = \rho|_{t=0} = 0. \end{cases} \tag{3.31}$$

Fix $j \geq -1$. I handle the density equation in (3.31) first. Applying the Δ_j operator, I have

$$\Delta_j\dot{\rho} - \kappa\Delta_j\Delta\rho = -(S_{j-2}u_1, \nabla)\Delta_j\rho + R_j(u_1, \rho) + (S_{j-2}v, \nabla)\Delta_j\rho_2 + R_j(v, \rho_2).$$

I use the fact that $\rho(0) = \rho_1(0) - \rho_2(0) = 0$, (3.12) for $j \geq 0$ and (3.11) for $j = -1$ to estimate

$$\begin{aligned} \|\Delta_j\rho(t)\|_\infty \leq \int_0^t (&\|R_j(u_1, \rho)\|_\infty \\ &+ \|(S_{j-2}u_1, \nabla)\Delta_j\rho\|_\infty + \|R_j(v, \rho_2)\|_\infty \\ &+ \|(S_{j-2}v, \nabla)\Delta_j\rho_2\|_\infty) \mathrm{d}\tau. \end{aligned} \tag{3.32}$$

It therefore suffices to bound the four terms on the right hand side. By theorem 2.3.3, one has for fixed j

$$\|R_j(u_1,\rho)\|_\infty \le C \sum_{|j-l|\le M_0} \{\|S_{l-2}\rho\|_\infty \|\Delta_l \nabla u_1\|_\infty + \|S_{l-2}\nabla u_1\|_\infty \|\Delta_l \rho\|_\infty\} + C2^j \sum_{l\ge j-M_0} {\sum_{|l-m|\le 1}}' 2^{-l} \|\Delta_l \nabla u_1\|_\infty \|\Delta_m \rho\|_\infty . \tag{3.33}$$

Let A_1, A_2 be the sum of the first and second lines above from $j=-1$ to N, respectively. (I will determine $N \ge 1$ later.) Then one can write

$$\begin{aligned}
A_1 &= C \sum_{j=-1}^{N} \sum_{|j-l|\le M_0} \{\|S_{l-2}\rho\|_\infty \|\Delta_l \nabla u_1\|_\infty + \|S_{l-2}\nabla u_1\|_\infty \|\Delta_l \rho\|_\infty\} \\
&\le C \left(\sup_{-1\le l\le N+M_0} \|S_{l-2}\rho\|_\infty \right) \sum_{l=-1}^{N+M_0} \|\Delta_l \nabla u_1\|_\infty \\
&\qquad + C \left(\sup_{-1\le l\le N+M_0} \|S_{l-2}\nabla u_1\|_\infty \right) \sum_{l=-1}^{N+M_0} \|\Delta_l \rho\|_\infty \\
&\le C \left(\sum_{l=-1}^{N+M_0} \|\Delta_l \nabla u_1\|_\infty \right) \left(\sum_{l=-1}^{N+M_0} \|\Delta_l \rho\|_\infty \right).
\end{aligned}$$

Next, I wish to control this term using only Π and a priori controlled quantities. Recall that $\Pi(\xi) \ge 1$ for all ξ. Then by Bernstein's inequality, $\omega_1 \in B_\Pi$ and proposition 2.1.6, I have

$$\begin{aligned}
\sum_{l=-1}^{N+M_0} \|\Delta_l \nabla u_1\|_\infty &\le \|\Delta_{-1} \nabla u_1\|_\infty + C \sum_{l=0}^{N+M_0} \|\Delta_l \omega_1\|_\infty \\
&\le C \|\Delta_{-1} \nabla u_1\|_{p_0} + C\Pi(N+M_0) \|\omega_1\|_\Pi \\
&\le C \|\Delta_{-1} \omega_1\|_{p_0} + C\Pi(N+M_0) \|\omega_1\|_\Pi \\
&\le C \|\omega_1\|_{p_0} + C\Pi(N+M_0) \|\omega_1\|_\Pi \\
&\le C \|\omega_1\| \,\Pi(N+M_0),
\end{aligned}$$

where $\|\cdot\| = \|\cdot\|_{B_\Pi \cap \mathrm{L}^{p_0}}$. Combined with the previous estimate, I conclude

$$A_1 \le C \|\omega_1\| \,\Pi(N+M_0) \sum_{l=-1}^{N+M_0} \|\Delta_l \rho\|_\infty . \tag{3.34}$$

For the sum of the second term in (3.33) from $j=-1$ to N, I use combinatorics and properties of Π to write:

$$\begin{aligned}
A_2 &= C \sum_{j=-1}^{N} \sum_{l \ge j-M_0} {\sum_{|l-m|\le 1}}' 2^{j-l} \|\Delta_l \nabla u_1\|_\infty \|\Delta_m \rho\|_\infty \\
&\le C \sum_{m=-1}^{\infty} \|\Delta_m \rho\|_\infty \left(\sum_{j=-1}^{\min(N,m+M_0+1)} 2^{j-m} \right) {\sum_{|l-m|\le 1}}' \|\Delta_l \nabla u_1\|_\infty \\
&\le C \sum_{m=-1}^{\infty} \left(2^{\min(N,m)-m} {\sum_{|l-m|\le 1}}' \|\Delta_l \nabla u_1\|_\infty \right) \|\Delta_m \rho\|_\infty \\
&= \sum_{m=-1}^{N} \left({\sum_{|l-m|\le 1}}' \|\Delta_l \nabla u_1\|_\infty \right) \|\Delta_m \rho\|_\infty \\
&\quad + \sum_{m=N+1}^{\infty} \left(2^{N-m} {\sum_{|l-m|\le 1}}' \|\Delta_l \nabla u_1\|_\infty \right) \|\Delta_m \rho\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\leq C\Pi(N)\left\|\omega_1\right\| \sum_{m=-1}^{N} \left\|\Delta_m \rho\right\|_\infty \\
&\qquad + C \sum_{m=N+1}^{\infty} \left(2^{N-m}\Pi(m)\left\|\omega_1\right\|\right) \left\|\Delta_m \rho\right\|_\infty \\
&\leq C\Pi(N)\left\|\omega_1\right\| \sum_{m=-1}^{N} \left\|\Delta_m \rho\right\|_\infty \\
&\qquad + C \sum_{m=N+1}^{\infty} \left(\int_N^\infty 2^{N-\xi}\Pi(\xi)\mathrm{d}\xi \left\|\omega_1\right\|\right) \left\|\Delta_m \rho\right\|_\infty \\
&\leq C\Pi(N)\left\|\omega_1\right\| \sum_{m=-1}^{\infty} \left\|\Delta_m \rho\right\|_\infty .
\end{aligned}
$$

Combined with the estimate for A_1, one concludes that

$$
\sum_{j=-1}^{N} \left\|R_j(u_1,\rho)\right\|_\infty \leq C\left\|\omega_1\right\| \Pi(N) \sum_{j=-1}^{\infty} \left\|\Delta_j \rho\right\|_\infty . \tag{3.35}
$$

Next, I estimate $\left\|R_j(v,\rho_2)\right\|_\infty$. In an identical argument to $R_j(u_1,\rho)$ (with u_1 in the place of ω_1 and ρ in place of v), I split the estimate into two terms:

$$
\begin{aligned}
\left\|R_j(v,\rho_2)\right\|_\infty \leq C \sum_{|j-l|\leq M_0} &\left\{\left\|S_{l-2}v\right\|_\infty \left\|\Delta_l \nabla\rho_2\right\|_\infty + \left\|S_{l-2}\nabla\rho_2\right\|_\infty \left\|\Delta_l v\right\|_\infty\right\} \\
&+ \ C2^j \sideset{}{'}\sum_{\substack{l\geq j-M_0 \\ |l-m|\leq 1}} 2^{-l} \left\|\Delta_l \nabla v\right\|_\infty \left\|\Delta_m \rho_2\right\|_\infty
\end{aligned}
$$

and conclude that

$$
\sum_{j=-1}^{N} \left\|R_j(v,\rho_2)\right\|_\infty \leq C\left\|\nabla\rho_2\right\| \Pi(N) \sum_{j=-1}^{\infty} \left\|\Delta_j v\right\|_\infty . \tag{3.36}
$$

Finally, I estimate $\left\|(S_{j-2}v,\nabla)\Delta_j \rho_2\right\|_\infty$ and $\left\|(S_{j-2}u_1,\nabla)\Delta_j \rho\right\|_\infty$. One has $\|(S_{j-2}v,\nabla)\Delta_j\rho_2\|_\infty \le \|S_{j-2}v\|_\infty \|\Delta_j\nabla\rho_2\|_\infty$, from which the estimate

$$\begin{aligned}\sum_{j=-1}^{N}\|(S_{j-2}v,\nabla)\Delta_j\rho_2\|_\infty &\le \left(\sup_{-1\le j\le N}\|S_{j-2}v\|_\infty\right)\sum_{j=-1}^{N}\|\Delta_j\nabla\rho_2\|_\infty \\ &\le C\,\|\nabla\rho_2\|\,\Pi(N)\sum_{j=-1}^{N}\|\Delta_j v\|_\infty\end{aligned} \tag{3.37}$$

easily follows. Similarly, one can write

$$\sum_{j=-1}^{N}\|(S_{j-2}u_1,\nabla)\Delta_j\rho\|_\infty \le C\,\|\omega_1\|\,\Pi(N)\sum_{j=-1}^{N}\|\Delta_j\rho\|_\infty. \tag{3.38}$$

Combining (3.35) and (3.36)-(3.38), I sum (3.32) from $j=-1$ to N and estimate it as:

$$\sum_{j=-1}^{N}\|\Delta_j\rho(t)\|_\infty \le C\Pi(N)\int_0^t\sum_{j=-1}^{\infty}\left(\|\Delta_j\rho(\tau)\|_\infty + \|\Delta_j v(\tau)\|_\infty\right)\mathrm{d}\tau, \tag{3.39}$$

where I have used (3.27) to bound $\|\omega_1(\tau)\|$, $\|\nabla\rho_2(\tau)\|$ uniformly on $[0,T]$.

Next, I apply the Δ_j operator to $\dot{v}$ and find

$$\begin{aligned}\Delta_j\dot{v} + (S_{j-2}u_1,\nabla)\Delta_j v = &-R_j(u_1,v) - (S_{j-2}v,\nabla)u_2 \\ &- R_j(v,u_2) - \Delta_j\nabla P + \begin{pmatrix} 0 \\ \Delta_j\rho\end{pmatrix}.\end{aligned} \tag{3.40}$$

In this setting, the velocity field evolving the function v is not v itself, but instead $S_{j-2}u_1$ and so one must consider the flow mappings $\{X_j(x,t;\tau)\}$ given by:

$$\begin{cases} \dot{X}_j(x,t;\tau) = S_{j-2}u_1(X_j(x,t;\tau),t) \\ X_j(x,0;\tau) = x(\tau)\end{cases}$$

(where S_{j-2} is S_{-1} when $j=-1,0$). Integrating (3.40) along $\{X_j(\alpha,t;\tau)\}$ and taking the L^∞-norm of both sides yields

$$\begin{aligned}\|\Delta_j v(t)\|_\infty \le \int_0^t &\|R_j(u_1,v)\|_\infty + \|(S_{j-2}v,\nabla)u_2\|_\infty \\ &+ \|R_j(v,u_2)\|_\infty + \|\Delta_j \nabla P\|_\infty + \|\Delta_j \rho\|_\infty \,\mathrm{d}\tau.\end{aligned} \tag{3.41}$$

The estimates for the first three terms are identical to estimates (3.35), (3.37) and (3.36), respectively, for the density equation, and the arguments will therefore not be repeated. Explicitly, we have:

$$\sum_{j=-1}^{N} \|R_j(u_1,v)\|_\infty \le C\,\|\omega_1\|\,\Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j v\|_\infty\,, \tag{3.42}$$

$$\sum_{j=-1}^{N} \|(S_{j-2}v,\nabla)\Delta_j u_2\|_\infty \le C\,\|\omega_2\|\,\Pi(N) \sum_{j=-1}^{N} \|\Delta_j v\|_\infty\,, \tag{3.43}$$

$$\sum_{j=-1}^{N} \|R_j(v,u_2)\|_\infty \le C\,\|\omega_2\|\,\Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j v\|_\infty\,. \tag{3.44}$$

For the pressure term, I take the divergence of (3.40) and use (3.29) to find:

$$\begin{aligned}\Delta_j \Delta P = &-\mathrm{div} R_j(u_1,v) - \mathrm{div} R_j(v,u_2) - \mathrm{tr}(\nabla \Delta_j v \cdot \nabla S_{j-2}u_1) \\ &- \mathrm{tr}(\nabla \Delta_j u_2 \cdot \nabla S_{j-2} v) + \Delta_j \partial_2 \rho.\end{aligned} \tag{3.45}$$

I consider two cases, $j \ge 0$ and $j=-1$. For $j \ge 0$, observe first that

$$-\nabla \Delta_j P(x) = \mathcal{F}_{\xi\to x}(i\xi|\xi|^{-2}(\Delta_j \Delta P)\hat{\ }(\xi)).$$

Using the estimate for $\Delta_j \Delta P$ above, as well as a Littlewood-Paley argument and Bernstein's inequality gives:

$$
\begin{aligned}
\|\Delta_j \nabla P\|_\infty \leq C \big(&\|R_j(u_1, v)\|_\infty + \|R_j(v, u_2)\|_\infty \\
&+ 2^{-j} \|\nabla \Delta_j v\|_\infty \|S_{j-2} \nabla u_1\|_\infty \\
&+ 2^{-j} \|\nabla \Delta_j u_2\|_\infty \|S_{j-2} \nabla v\|_\infty + \|\Delta_j \rho\|_\infty\big) \\
\leq C \big(&\|R_j(u_1, v)\|_\infty + \|R_j(v, u_2)\|_\infty + \|\Delta_j v\|_\infty \|S_{j-2} \nabla u_1\|_\infty \\
&+ \|\Delta_j u_2\|_\infty \|S_{j-2} \nabla v\|_\infty + \|\Delta_j \rho\|_\infty\big) .
\end{aligned}
\tag{3.46}
$$

For $j = -1$, we use

$$
\begin{aligned}
\|\Delta_{-1} \nabla P\|_\infty &\leq C \|\Delta_{-1} \nabla P\|_{p_2} \\
&\leq C \left\| - \sum_{k=1}^{n} \partial_k \Delta_{-1} \{u_1^{(k)} v + v^{(k)} u_2\} \right\|_{p_2} + \|\Delta_{-1} \rho\|_{p_2} \\
&\leq C \|\Delta_{-1}(u_1 \otimes v)\|_{p_2} + C \|\Delta_{-1}(v \otimes u_2)\|_{p_2} + \|\Delta_{-1} \rho\|_{p_2} \\
&\leq C \sum_{|l-m| \leq M_0} \Big(\|\Delta_l u_1\|_{p_2} + \|\Delta_l u_2\|_{p_2}\Big) \|\Delta_m v\|_\infty + \|\Delta_{-1} \rho\|_{p_2} ,
\end{aligned}
\tag{3.47}
$$

where $p_2 \in [\frac{np_0}{n-p_0}, \infty)$. Sobolev embedding and (3.27) then gives the desired control over pressure in terms of the other members of the right hand side of (3.41).

Combining these estimates with those for the previous three terms and using (3.27), one finds:

$$
\sum_{j=-1}^{N} \|\Delta_j v(t)\|_\infty \leq C\Pi(N) \int_0^t \sum_{j=-1}^{\infty} (\|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty) \mathrm{d}\tau, \tag{3.48}
$$

and therefore:

$$\sum_{j=-1}^{N} \|\Delta_j v(t)\|_\infty + \|\Delta_j \rho(t)\|_\infty \leq \qquad (3.49)$$
$$C\Pi(N) \int_0^t \sum_{j=-1}^{\infty} \|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty \, \mathrm{d}\tau.$$

Ideally, I wish to use a Gronwall-type estimate for the above equation, but the quantities on the left hand side and the integrand are not the same. With this in mind, I must control the tail of the $B^0_{\infty,1}$-norms of density and velocity as well - in order to apply Gronwall's inequality to (3.49), I need to first estimate

$$\sum_{j=N+1}^{\infty} \|\Delta_j v(t)\|_\infty + \|\Delta_j \rho(t)\|_\infty .$$

Control of these two quantities is nearly identical, so I address only $\rho(t)$. (In the argument that follows, I suppress time to avoid cluttered notation.) Using Bernstein's inequality, one can write

$$\sum_{j=N+1}^{\infty} \|\Delta_j \rho\|_\infty \leq C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla \rho\|_\infty .$$

Next set $d_k = \sum_{j=-1}^{k} \|\Delta_j \nabla\rho\|_\infty$, and use Abel's lemma to write

$$\begin{aligned} \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla \rho\|_\infty &\leq \sum_{j=N+1}^{\infty} 2^{-j}(d_j - d_{j-1}) \qquad (3.50) \\ &\leq -2^{(N-1)} d_N + \sum_{j=N+1}^{\infty} d_j (2^{-j} - 2^{-(j+1)}) \\ &\leq -2^{(N-1)} d_N + \|\nabla \rho\|_\Pi \sum_{j=N+1}^{\infty} 2^{-j}\Pi(j) \\ &\leq -2^{(N-1)} d_N + \|\nabla \rho\|_\Pi \int_{j=N+1}^{\infty} 2^{-j}\Pi(j)\mathrm{d}j \\ &\leq C 2^{-N}\Pi(N), \end{aligned}$$

where I have used conditions (ii) and (iii) on Π and (3.27). This allows me to write (3.49) as

$$\sum_{j=-1}^{\infty}\left(\|\Delta_j v\|_\infty + \|\Delta_j \rho\|_\infty\right) \le \sum_{j=-1}^{N}\left(\|\Delta_j v\|_\infty + \|\Delta_j \rho\|_\infty\right) + C2^{-N}\Pi(N). \quad (3.51)$$

Next, define the function $F(t)$ by

$$F(t) = \int_0^t \sum_{j=-1}^{\infty}\left(\|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty\right) \mathrm{d}\tau.$$

Then one can use (3.51) to achieve the estimate

$$F'(t) \le C\Pi(N)F(t) + C2^{-N}\Pi(N). \quad (3.52)$$

By construction, $F(t)$ is a monotonically nondecreasing, absolutely continuous function, and the differential form of Gronwall's inequality shows that one must have $\|F'\|_{\mathrm{L}^\infty([0,T])} \le C$. Since $F(0) = 0$ by construction, this implies that there exists some t_0 such that

$$F(t) \equiv 0 \text{ on } [0, t_0], \qquad F(t) > 0 \text{ on } (t_0, T].$$

If $t_0 = T$, then I have uniqueness. Therefore, by way of contradiction, I assume that $t_0 < T$. Fix $\varepsilon > 0$ sufficiently small that $t_0 + \varepsilon < T$ and $F(t) < 2^{-M_1-1}$ on $(t_0, t_0 + \varepsilon)$ (where M_1 will be determined later). Choose $t \in (t_0, t_0 + \varepsilon)$ and let

$N = \max\{1, \lceil -\log_2 F(t)\rceil\}$. Then (3.52) becomes

$$F'(t) \le C\Pi(-\log_2 F(t))F(t), \quad F(0) = 0. \quad (3.53)$$

It is here that I make use of the growth condition on Π given by (3.25). From this assumption and a change of variables $(-\log_2 F) \mapsto \alpha$, one has

$$\int_0^{1/2} F^{-1}(\Pi(-\log_2 F))^{-1}\mathrm{d}F = C\int_1^{\infty}(\Pi(\alpha))^{-1}\mathrm{d}\alpha = \infty.$$

I can then apply theorem 2.1.5, the Osgood Uniqueness Theorem, to the ODE

$$\begin{cases} \dot{\eta}(t,\delta) = C\Pi(-\log_2 \eta)\eta \\ \eta(t_0,\delta) = \delta \end{cases} \tag{3.54}$$

and show that for $\delta > 0$ sufficiently small, a unique solution to (3.54) exists on $(t_0, t_0+\varepsilon)$, and depends continuously on δ. Furthermore, one has

$$F(t) < \eta(t,\delta) \text{ for all } t \in [t_0, t_0+\varepsilon). \tag{3.55}$$

I prove this strict inequality by contradiction. Suppose there exists some t_1 such that $t_1 = \min_{[t_0,t_0+\varepsilon)}\{\, t \mid F(t) = \eta(t,\delta)\}$. Then since $\delta > 0$, we must have

$$\begin{aligned} F(t_1) &\le \int_{t_0}^{t_1} \Pi(-\log_2 F(\tau))F(\tau)\mathrm{d}\tau \\ &\le \int_{t_0}^{t_1} \Pi(-\log_2 \eta(\tau,\delta))\eta(\tau,\delta)\mathrm{d}\tau \\ &< \delta + \int_{t_0}^{t_1} \Pi(-\log_2 \eta(\tau,\delta))\eta(\tau,\delta)\mathrm{d}\tau \\ &= \eta(t_1,\delta), \end{aligned}$$

contradicting the definition of t_1. Therefore (3.55) holds. Finally, I choose M_1 such that (3.26) holds for $\alpha \ge M_1$. As $\delta \to 0^+$, one must have that $F \equiv 0$ on $[t_0, t_0+\varepsilon)$, contradicting the definition of t_0. This implies that $t_0 = T$, and uniqueness is proven. $\square$

3.3 Construction of the flow

In the final section of this chapter, I construct a solution (u, ρ) to the Boussinesq equations and show that the velocity (resp., density) is weak-* continuous in time with values in B_{Γ_1} (resp., B_Γ). I do so by using an approximation argument that follows a similar approach to that of the proof of uniqueness in the previous section. The added challenge is that I can no longer assume the initial data of the system I am bounding is identically zero, as it was in the case of uniqueness. To circumvent this, I use an Abel's lemma argument to show that the initial value term can be controlled in the B_Γ norm in such a way that I am still able to apply theorem 2.1.5 to the resulting ODE system. For the existence of Sobolev-regular solutions, I use a result of Dongho Chae's, stated as proposition 2.3.6 in chapter 2. Unlike section 3.2, I assume that Γ, Γ_1 satisfy (i)-(vi) of section 2.2. This section is dedicated to proving the following theorems:

Theorem 3.3.1. *For $1 < p_0 < 2 < p_1 < \infty$, let $f \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ and $g \in \mathrm{W}^{1,p_0} \cap \mathrm{W}^{1,p_1}$ such that $\nabla g \in B_\Gamma$. Assume that*

$$(\alpha + 2)\Gamma'(\alpha) \leq C \text{ for a.e. } \alpha \in [-1, \infty).$$

Then there exists a $T > 0$ (depending on Γ, f and g) and a solution (u, ρ) to the system of equations $(B_{\kappa,0})$ with $u = \mathcal{K} * \omega$, such that

$$\omega(\,\cdot\,) \in \mathrm{L}^\infty([0, T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0, T]; B_{\Gamma_1}), \tag{3.56}$$

$$\nabla\rho(\,\cdot\,) \in \mathrm{L}^\infty([0, T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0, T]; B_\Gamma). \tag{3.57}$$

Theorem 3.3.2. *Let f and g be as in theorem 3.3.1. Assume that*

$$\Gamma'(\alpha)\Gamma_1(\alpha) \leq C \text{ for a.e. } \alpha \in [-1,\infty).$$

Then there exist (u,ρ) solving $(B_{\kappa,0})$ such that

$$\omega(\,\cdot\,) \in \mathrm{L}^\infty_{loc}([0,\infty);\mathrm{L}^{p_0}\cap\mathrm{L}^{p_1}) \cap C_{w^*}([0,\infty);B_{\Gamma_1}), \tag{3.58}$$

$$\nabla\rho(\,\cdot\,) \in \mathrm{L}^\infty_{loc}([0,\infty);\mathrm{L}^{p_0}\cap\mathrm{L}^{p_1}) \cap C_{w^*}([0,\infty);B_{\Gamma}). \tag{3.59}$$

It should be noted that $C_{w^*}([0,T];B_{\Gamma_1})$ is the space of weak-* continuous functions with values in B_{Γ_1} in the sense of duality, $(H_{\Gamma_1})' = B_{\Gamma_1}$. Recall from section 2.3 that the dual of B_{Γ_1} is isomorphic to the space H_{Γ_1} given by:

$$H_{\Gamma_1} = \left\{ f \in \mathcal{S}' \;\middle|\; \exists\, \{d_j\}_{j=-1}^{\infty},\ d_j \geq 0,\ \sum_{j=-1}^{\infty} d_j < \infty \right.$$
$$\left. \text{and } \|\Delta_m f\|_1 \leq \sum_{j\geq m} d_j \Gamma_1(j)^{-1}\ \forall\, m \geq -1. \right\}$$

Furthermore, recall from proposition 2.3.2 that for $f \in \mathcal{S}'$, the H^r Sobolev norm is equivalent to the Littlewood-Paley decomposition:

$$\|\Delta_{-1} f\|_2 + \left(\sum_{k=0}^{\infty} 2^{2kr} \|\Delta_k f\|_2^2 \right)^{\frac{1}{2}}$$

A simple application of the Bernstein and Cauchy-Schwarz inequality gives:

Proposition 3.3.3. *For $r > 1$, $H^r(\mathbb{R}^2) \subset B_\Gamma(\mathbb{R}^2)$.*

Proof. Fix $N \geq -1$, and suppose that $f \in H^r(\mathbb{R}^2)$. Then

$$\begin{aligned}
\sum_{j=-1}^{N} \|\Delta_j f\|_\infty &\leq C \sum_{j=-1}^{N} 2^j \|\Delta_j f\|_2 \\
&\leq C \sum_{j=-1}^{N} (2^{jr} \|\Delta_j f\|_2) 2^{-j(r-1)} \\
&\leq C \left(\sum_{j=-1}^{\infty} 2^{2jr} \|\Delta_j f\|_2 \right) \left(\sum_{j=-1}^{N} 2^{-2j(r-1)} \right) \\
&\leq C \|f\|_{H^r} \left(\sum_{j=-1}^{N} 2^{-2j(r-1)} \right) \\
&\leq C \|f\|_{H^r},
\end{aligned}$$

which is trivially less than $C\Gamma(N)$ for the proper choice of C since $\Gamma(\alpha) \geq 1$ for all α. □

Proof of theorem 3.3.1. For any $m \geq 1$, construct the solution (u_m, ρ_m) given by proposition 2.3.6 such that

$$\omega_m(0) = S_m f \in \bigcap_{r>2} H^r \quad \text{and} \quad \nabla\rho_m(0) = S_m \nabla g \in \bigcap_{r>2} H^r. \tag{3.60}$$

Since $\|S_m h\|_p \leq \|h\|_p$ for $p \in [1, \infty]$ and any $h \in \mathcal{S}'$ by definition of the S_m operator, one has

$$\|\omega_m(0)\|_{p_0}, \quad \|\nabla\rho_m(0)\|_{p_0} \leq C \text{ and } \|\omega_m(0)\|_{p_1}, \quad \|\nabla\rho_m(0)\|_{p_1} \leq C. \tag{3.61}$$

Furthermore, the definition of Δ_j and S_m give

$$\|\omega_m(0)\|_\Gamma \leq C \|f\|_\Gamma, \qquad \|\nabla\rho_m(0)\|_\Gamma \leq C \|\nabla g\|_\Gamma. \tag{3.62}$$

Combined with propositions 2.3.6 and 3.3.3 one concludes that

$$\begin{aligned}&\omega_m(\cdot) \in \mathrm{L}^\infty_{loc}([0,\infty); B_{\Gamma_1}) \\ &\nabla\rho_m(\cdot) \in \mathrm{L}^\infty_{loc}([0,\infty); B_\Gamma).\end{aligned} \tag{3.63}$$

Using theorem 3.1.1 along with (3.61), (3.62) and (3.63) I conclude that there is a $T > 0$ such that

$$\begin{aligned}&\omega_m(\cdot), \nabla\rho_m(\cdot) \in \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}), \\ &\omega_m(\cdot) \in \mathrm{L}^\infty([0,T]; B_{\Gamma_1}), \\ &\nabla\rho_m(\cdot) \in \mathrm{L}^\infty([0,T]; B_\Gamma).\end{aligned} \tag{3.64}$$

Fix two indices, m and l. Then set

$$\begin{array}{ll} u_{\{m,l\}} = \mathcal{K} * \omega_{\{m,l\}}, & \omega = \omega_m - \omega_l \\ v = u_m - u_l, & \rho = \rho_m - \rho_l. \end{array} \tag{3.65}$$

I use the same estimate as in the uniqueness proof of section 3.2, only in this case I replace Π with Γ_1, and I cannot assume that $\Delta_j v(0)$ and $\Delta_j\rho(0)$ are zero. To wit, one has for any $N \geq 1$,

$$\begin{aligned}&\sum_{j=-1}^{N} \left(\|\Delta_j v(t)\|_\infty + \|\Delta_j \rho(t)\|_\infty\right) \\ &\qquad \leq \sum_{j=-1}^{N} \left(\|\Delta_j v(0)\|_\infty + \|\Delta_j \rho(0)\|_\infty\right) \\ &\qquad\quad + C\Gamma_1(N) \int_0^t \sum_{j=-1}^{\infty} (\|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty) \mathrm{d}\tau \\ &\qquad\quad + C2^{-N}\Gamma_1(N).\end{aligned} \tag{3.66}$$

As in the previous section, I define

$$F(t) = \int_0^t \sum_{j=-1}^{\infty} \left(\|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty \right) \mathrm{d}\tau,$$

which, for fixed $t \in [0, T]$ and $N = \max\{1, \lceil -\log_2 F(t) \rceil\}$ allows me to write (3.66) as

$$F'(t) \leq \sum_{j=-1}^{N} \left(\|\Delta_j v(0)\|_\infty + \|\Delta_j \rho(0)\|_\infty \right) + C\Gamma_1(-\log_2 F(t))F(t). \tag{3.67}$$

Denote the sum on the right hand side by

$$\kappa_{m,l} + \iota_{m,l} := \sum_{j=-1}^{N} \|\Delta_j v(0)\|_\infty + \sum_{j=-1}^{N} \|\Delta_j \rho(0)\|_\infty .$$

The bounds on $\kappa_{m,l}$ and $\iota_{m,l}$ are quite similar so I demonstrate only the latter. To bound $\iota_{m,l}$, one first writes

$$\begin{aligned}
\iota_{m,l} &= \sum_{j=-1}^{N} \|\Delta_j \rho(0)\|_\infty \\
&\leq \sum_{j=-1}^{N} 2^{-j} \|\Delta_j (S_m - S_l) g\|_\infty \\
&= \sum_{j=-1}^{N} 2^{-j} \left\| \Delta_j \Big(\sum_{k=l+1}^{m} \Delta_k g \Big) \right\|_\infty \\
&\leq \sum_{k=l+1}^{m} \sum_{|k-j| \leq M_0} 2^{-k} \|\Delta_j \Delta_k g\|_\infty \\
&\leq C \sum_{k=l+1}^{\infty} 2^{-k} \|\Delta_k g\|_\infty .
\end{aligned}$$

Using an Abel's Lemma argument identical to (3.50), I conclude that

$$\iota_{m,l} \leq C 2^{-l} \Gamma(l). \tag{3.68}$$

Next, I integrate equation (3.67) in time. Using the estimate (3.68) and the analogous estimate for $\kappa_{m,l}$, one can write

$$F(t) \leq C2^{-l}\Gamma(l) + C\int_0^t \Gamma_1(-\log_2 F(\tau))F(\tau)\mathrm{d}\tau.$$

Let η solve the ODE:

$$\begin{cases} \dot{\eta} = C\Gamma_1(-\log_2 \eta)\eta \\ \eta(0) = C2^{-l}\Gamma(l). \end{cases}$$

Then a simple Gronwall argument gives

$$F(t) \leq \eta(t, C2^{-l}\Gamma(l))$$

for $t \in [0,T]$. Combined with (3.67), one has

$$F'(t) \leq C2^{-l}\Gamma(l) + C\Gamma_1[-\log_2 \eta(t, C2^{-l}\Gamma(l))]\eta(t, C2^{-l}\Gamma(l)) \tag{3.69}$$

for all $t \in [0,T]$. Since $\|F'\|_{\mathrm{L}^\infty([0,T])} \leq C$, one must have that $\{u_m\}$ and $\{\rho_m\}$ are Cauchy sequences in the Banach space $\mathrm{L}^\infty([0,T]; B^0_{\infty,1})$. Therefore, there exists u, ρ in this space such that:

$$u_m \longrightarrow u, \quad \rho_m \longrightarrow \rho \in \mathrm{L}^\infty([0,T]; B^0_{\infty,1}). \tag{3.70}$$

As I will show next, (3.70) in fact implies that for $\omega = \text{curl } u$,

$$\|\omega\|_{\Gamma_1}, \|\nabla\rho\|_{\Gamma} \in \mathrm{L}^\infty([0,T]). \tag{3.71}$$

Define the seminorm ν_N on $\mathrm{L}^\infty([0,T]; B^0_{\infty,1})$ given by

$$\nu_N(f) = \left\| \sum_{j=-1}^{N} \|\Delta_j f(\cdot)\|_\infty \right\|_{\mathrm{L}^\infty([0,T])}.$$

By (3.70), it is clear that $\nu_N(u_m - u)$ and $\nu_N(\rho_m - \rho)$ tend to zero as $m \to \infty$. Using Bernstein's inequality, one has

$$\begin{aligned}
\left\| \sum_{j=-1}^{N} \|\Delta_j(\omega_m - \omega)\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} &\leq \left\| \sum_{j=-1}^{N} \|2^j \Delta_j(u_m - u)\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} \\
&\leq C2^N \left\| \sum_{j=-1}^{N} \|\Delta_j(u_m - u)\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} \\
&= C2^N \nu_N(u_m - u),
\end{aligned}$$

and similarly for $\nabla\rho_m$. In the case of the latter, this yields

$$\begin{aligned}
&\left\| \sum_{j=-1}^{N} \|\Delta_j \nabla\rho_m\|_\infty - \|\Delta_j \nabla\rho\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} \qquad (3.72) \\
&\qquad \leq \left\| \sum_{j=-1}^{N} \left| \|\Delta_j \nabla\rho_m\|_\infty - \|\Delta_j \nabla\rho\|_\infty \right| \right\|_{\mathrm{L}^\infty([0,T])} \\
&\qquad \leq \left\| \sum_{j=-1}^{N} \|\Delta_j(\nabla\rho_m - \nabla\rho)\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} \\
&\qquad \leq C2^N \nu_N(\rho_m - \rho).
\end{aligned}$$

By (3.64), I have that $\sum_{j=-1}^{N} \|\Delta_j \nabla\rho_m(t)\|_\infty \leq C\Gamma(N)$, where C is independent of our choice of m. Using this fact and (3.72), one then computes

$$\begin{aligned}
\left\| \sum_{j=-1}^{N} \|\Delta_j \nabla\rho\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} &\leq \left\| \sum_{j=-1}^{N} \|\Delta_j \nabla\rho\|_\infty - \|\Delta_j \nabla\rho_m\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} \\
&\quad + \left\| \sum_{j=-1}^{N} \|\Delta_j \nabla\rho_m\|_\infty \right\|_{\mathrm{L}^\infty([0,T])} \\
&\leq C2^N \nu_N(\rho_m - \rho) + C\Gamma(N).
\end{aligned}$$

Finally, passing to the limit as $m \to \infty$ gives the second inclusion of (3.71). The bound on ω is identical (with Γ_1 in place of Γ), and is therefore omitted.

It remains to show that (u, ρ) satisfy the Boussinesq equations. Note that since $\{u_m\}$ and $\{\rho_m\}$ are Cauchy sequences in $C([0,T]; B^0_{\infty,1})$ (in addition to $\mathrm{L}^\infty([0,T]; B^0_{\infty,1})$), I can conclude that

$$u_m \longrightarrow u, \rho_m \longrightarrow \rho \text{ in } \mathrm{L}^\infty([0,T] \times \mathbb{R}^2) \cap C([0,T] \times \mathbb{R}^2). \tag{3.73}$$

Up to choosing a subsequence, the above convergence yields:

$$\omega_m \overset{w*}{\rightharpoonup} \omega, \;\; \nabla\rho_m \overset{w*}{\rightharpoonup} \nabla\rho \text{ in } \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0}) \cap \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_1}), \tag{3.74}$$

$$\dot{u}_m \overset{w*}{\rightharpoonup} \dot{u}, \;\; \dot{\rho}_m \overset{w*}{\rightharpoonup} \dot{\rho} \text{ in } \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0}) \cap \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_1}). \tag{3.75}$$

I wish to show that the limiting functions (u, ρ) are weak solutions to $(B_{\kappa,0})$. Let $\beta \in \mathcal{S}$, div $\beta = 0$ be a test function, and let $\theta \in \mathcal{D}([0,T])$. By definition of (u_m, ρ_m), one finds

$$\begin{aligned}
\langle u_m(0), \beta\rangle\theta(0) + \int_0^T & \langle u_m(\tau), \beta\rangle\dot{\theta}(\tau) \\
& + \langle u_m(\tau), (u_m(\tau), \nabla)\beta\rangle\theta(\tau) - \langle \rho_m(\tau), \beta\rangle\theta(\tau)\mathrm{d}\tau = 0, \\
\langle \rho_m(0), \beta\rangle\theta(0) + \int_0^T & \langle \rho_m(\tau), \beta\rangle\dot{\theta}(\tau) \\
& + \langle \rho_m(\tau), (u_m(\tau), \nabla)\beta\rangle\theta(\tau) + \kappa\langle \rho_m(\tau), \Delta\beta\rangle\theta(\tau)\mathrm{d}\tau = 0.
\end{aligned}$$

From (3.60) and the definition of ω_m, ρ_m, one has

$$\begin{aligned}
\langle u_m(0), \beta\rangle &\longrightarrow \langle \mathcal{K} * f, \beta\rangle \\
\langle \rho_m(0), \beta\rangle &\longrightarrow \langle g, \beta\rangle.
\end{aligned}$$

Sending $m \to \infty$ and utilizing (3.73)-(3.75), I conclude that (u, ρ) solve $(B_{\kappa,0})$ weakly.

It remains to show that $\omega(\cdot), \nabla\rho(\cdot)$ are weak-* continuous with values in B_{Γ_1} and B_{Γ}, respectively. Since the proofs are nearly identical up to our choice of target space, I consider only the case of $\nabla\rho$. $\{\rho_m\}$ is a Cauchy sequence in $C([0,T]; B^0_{\infty,1})$, therefore one has that

$$\|\nabla\rho - \nabla\rho_m\|_{C([0,T];B^{-1}_{\infty,1})} \longrightarrow 0$$

as $m \to \infty$. Fix $h \in H_\Gamma$. Define the mapping $\pi(t) := \langle \nabla\rho(t), h\rangle$ for $t \in [0,T]$, and define $\pi_m(t)$ similarly for $\rho_m(t)$. For any $t_0 \in [0,T]$,

$$\pi(t) - \pi(t_0) = (\pi - \pi_m)(t) - (\pi - \pi_m)(t_0) + (\pi_m(t) - \pi_m(t_0)). \tag{3.76}$$

By (3.63), I have that for fixed m, $\pi_m(t) - \pi_m(t_0) \to 0$ as $t \to t_0$. By construction, for any $\tilde{h} \in H_\Gamma$,

$$|(\pi - \pi_m)(t)| \le \left|\langle(\nabla\rho - \nabla\rho_m)(t), h - \tilde{h}\rangle\right| + \left|\langle(\nabla\rho - \nabla\rho_m)(t), \tilde{h}\rangle\right|.$$

Let $\|\cdot\|_{\Gamma'}$ be the norm of H_Γ, given by

$$\|f\|_{\Gamma'} = \inf_{\{d_j\}} \sum_{j=-1}^{\infty} d_j.$$

Fix $\delta > 0$, and consider the space $B^1_{1,1+\delta^{-1}}$. Any Besov Space based on the L^1-norm contains every L^1-function with bounded Fourier spectrum (since the Littlewood-Paley decomposition of such a function has only finitely many nonzero terms). Furthermore, these functions are dense in H_Γ. This implies that $B^1_{1,1+\delta^{-1}}$ is a dense subset of H_Γ, and I can choose $\tilde{h} \in B^1_{1,1+\delta^{-1}}$ such that $\left\|h - \tilde{h}\right\|_{\Gamma'} < \varepsilon$ for any $\varepsilon > 0$. Because $\|\nabla\rho(t)\|_\Gamma$ and $\|\nabla\rho_m(t)\|_\Gamma$ are uniformly bounded on $[0,T]$,

$$\left|\langle(\nabla\rho - \nabla\rho_m)(t), h - \tilde{h}\rangle\right| < C\varepsilon. \tag{3.77}$$

Finally, I use the duality $(B^1_{1,1+\delta^{-1}})' = B^{-1}_{\infty,1+\delta}$ and the embedding $B^{-1}_{\infty,1} \hookrightarrow B^{-1}_{\infty,1+\delta}$, to write

$$\left|\langle(\nabla\rho - \nabla\rho_m)(t), \tilde{h}\rangle\right| \leq C \left\|\nabla\rho - \nabla\rho_m\right\|_{C([0,T];B^{-1}_{\infty,1})} \left\|\tilde{h}\right\|_{B^1_{1,1+\delta^{-1}}}. \qquad (3.78)$$

By choosing m sufficiently large, I can make the right hand side of (3.78) less than ε. Combined with (3.76), this gives

$$\limsup_{t\to t_0} |\pi(t) - \pi(t_0)| \leq C\varepsilon,$$

which yields the desired result for $\nabla\rho$. The proof for ω is similar, and theorem 3.3.1 is proved. $\square$

Proof of theorem 3.3.2. The proof follows that of theorem 3.3.1, except that, as discussed in remark 3.1.1, the choice of $T > 0$ is arbitrary and no longer depends on Γ and Γ_1. Since the proof is identical to the above except in that respect, it is omitted. $\square$

Appendix

Appendix A

Volume-Preserving Homeomorphisms and B_Γ

In this appendix, I mention some of the elements used to prove theorem 3.2.1 and proposition 3.1.2. As Vishik shows in [31], one can represent the B_Γ spaces using standard wavelet expansions. I state without proof the following result:

Proposition A.0.4. *Let $f \in B_\Gamma$. Then there exist wavelet generating functions Ψ and ψ, as well as coefficients $\{a_{-1,k}\}$ and $\{a_{m,k}\}$ such that*

$$f(x) = \sum_{k \in \mathbb{Z}^n} a_{-1,k} \Psi(x-k) + \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}^n} a_{m,k} \psi_{m,k}(x).$$

This series is convergent in $\mathcal{S}'(\mathbb{R}^n)$, and there exists a constant C (independent of f) such that

$$C^{-1} \|f\|_\Gamma \leq \sup_{N \geq -1} \left(\sum_{m=-1}^{N} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \right) \Gamma(N)^{-1} \leq C \|f\|_\Gamma . \tag{A.1}$$

Let $f \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$, and let $X(t)$ be the flow map generated by $u(x,t)$, where u solves the Euler equations. The goal of the remainder of this appendix is to sketch a proof of the following result of Vishik ([31], proposition 5.4), which is analogous to proposition 3.1.2:

Proposition A.0.5. *The following estimates hold:*

1. *When* Γ *satisfies* $(\alpha+2)\Gamma'(\alpha) \leq C$ *for a.e.* $\alpha \in [-1,\infty)$, *there exists a* $T>0$ *such that*

$$\left\|f \circ X^{-1}(t)\right\|_{\Gamma_1} \leq \|f\|_{\Gamma} 2^{C\int_0^t \lambda(\tau)d\tau},$$

for $t \in [0,T]$, *where* C *depends on* T *and the initial data in theorem 3.1.1.* $\lambda(t)$ *is bounded above by a solution to the ODE*

$$\begin{cases} \partial_t \lambda_1 = C\lambda_1^2 \\ \lambda_1(0) = \max\{\|f\|_{\Gamma_1}, 1\} \end{cases} \tag{A.2}$$

and hence for T *less than the blowup time of (A.2),* $\lambda(t) \leq \lambda_1(t) \leq C(T)$.

2. *When* Γ *and* Γ_1 *satisfy* $\Gamma'(\alpha)\Gamma_1(\alpha) \leq C$ *for a.e.* $\alpha \in [-1,\infty)$, *we have*

$$\left\|f \circ X_u^{-1}(t)\right\|_{\Gamma_1} \leq C\|f\|_{\Gamma}\left(1+\int_0^t \lambda(\tau)d\tau\right) \tag{A.3}$$

where $\lambda(t)$ *is bounded above by a solution to the ODE*

$$\begin{cases} \partial_t \lambda_2 = C\lambda_2 \\ \lambda_2(0) = \max\{\|f\|_{\Gamma_1}, 1\} \end{cases} \tag{A.4}$$

and hence for any $t>0$, $\lambda(t) \leq \lambda_2(t) < \infty$.

Using standard Littlewood-Paley estimates, one has the following moduli of continuity for u (see chapter 3 of [31] for details):

$$|u(x,t)-u(y,t)| \leq C_0|x-y| \quad \text{for } |x-y| \geq \frac{1}{2},\ t \in [0,T], \tag{A.5}$$

$$|u(x,t)-u(y,t)| \leq C_0\lambda(t)\Gamma_1(-\log_2|x-y|)|x-y| \quad \text{for } |x-y| \leq \frac{1}{2}. \tag{A.6}$$

I wish to combine these two estimates, so I define a linear interpolation function $\tilde{\Gamma}_1(m,t)$, by

$$\tilde{\Gamma}_1(m,t) = \begin{cases} (\lambda(t))^{-1} & -\infty < m \leq m_1 \\ 1 + (m+1)\Gamma_1'(-1) & m_1 \leq m \leq -1 \\ \Gamma_1(m) & -1 < m < \infty, \end{cases} \tag{A.7}$$

where $m_1 = -1 - (1 - (\lambda(t))^{-1})(\Gamma_1'(-1))^{-1}$. By construction, this implies that

$$|u(x,t) - u(y,t)| \leq C_0 \lambda(t) \tilde{\Gamma}_1(-\log_2 |x-y|, t)|x-y| \tag{A.8}$$

whenever $x \neq y$.

Next, consider the Cauchy problem:

$$\begin{cases} \dot{\mu}(m,t) = -C_0(\log_2 e)\lambda(t)\tilde{\Gamma}_1(\mu(m,t),t) \\ \mu(m,0) = m \end{cases} \tag{A.9}$$

for all $m \in \mathbb{R}$. Using the Lipschitz continuity of $\tilde{\Gamma}_1$, a unique solution exists to (A.9) for all $t \in [0,T]$. I denote the solution by $\mu(m,t)$ and have the following result regarding the stretching of the flow field:

Proposition A.0.6. *Let $x, y \in \mathbb{R}^2$, $x \neq y$. Then for $m = -\log_2 |x-y|$ and $t \in [0,T]$,*

$$|X(t)x - X(t)y| \leq 2^{-\mu(m,t)}.$$

Remark A.0.1. A similar Cauchy problem and argument allows one to control the stretching of X^{-1} via a function I denote by $\eta(m,t)$. Since the results involving η and the inverse flow field follow closely those of X and μ, I omit them.

Proof. Let $\xi(t) = |X(t)x - X(t)y|$. Then (A.8) gives

$$\begin{aligned}\frac{d}{dt}\xi(t) &\le |u(X(t)x, t) - u(X(t)y, t)| \\ &\le C_0\lambda(t)\tilde{\Gamma}_1(-\log_2 \xi(t), t)\xi(t).\end{aligned}$$

This implies that

$$\frac{d}{dt}(-\log_2 \xi(t)) \ge -C_0\lambda(t)(\log_2 e)\tilde{\Gamma}_1(-\log_2 \xi(t), t).$$

The right hand side is a non-increasing function on $\mathbb{R}$ of $-\log_2 \xi(t)$, so it follows from a Gronwall-type inequality that:

$$-\log_2 \xi(t) \ge \mu(m, t)$$

for $t \in [0, T]$, which proves the proposition. □

A straight-forward application of convexity and Gronwall's inequality gives that for fixed $t \in [0, T]$, the map $m \mapsto \mu(m, t)$ is concave. The next step in proving Proposition A.0.5 is the following technical result, which constitutes the entirety of section 4 of [31] (and which I therefore state without proof):

Proposition A.0.7. *Let $X : \mathbb{R}^n \xrightarrow{onto} \mathbb{R}^n$ be a volume-preserving homeomorphism. Let σ, $\zeta : \mathbb{R} \to \mathbb{R}^+$ be decreasing functions such that $\lim_{\xi\to\infty} \sigma(\xi)$, $\zeta(\xi) = 0$ and assume*

$$|X^{-1}(x) - X^{-1}(y)| \le C_{X^{-1}}\sigma(-\log_2 |x - y|), \tag{A.10}$$

$$|X(x) - X(y)| \le C_X\zeta(-\log_2 |x - y|), \tag{A.11}$$

$$\sigma(\xi) = \zeta(\xi) = 2^{-\xi} \quad \text{for } \xi \le 0, \tag{A.12}$$

$$\log_2 \sigma(\xi), \ \ \log_2 \zeta(\xi) \quad \text{are convex.} \tag{A.13}$$

Let f be any function with the following wavelet decomposition:

$$
\begin{aligned}
&f_l(x) = \sum_{k\in\mathbb{Z}^n} a_{l,k}\psi(2^l x - k), \quad \textit{for } l \geq 0, \\
&f_{-1}(x) = \sum_{k\in\mathbb{Z}^n} a_{l,k}\Psi(2^l x - k), \\
&\sup_{k\in\mathbb{Z}^n} |a_{l,k}| \leq 1.
\end{aligned}
$$

Then there exists a constant $\gamma = \gamma(n)$ and a constant $C = C(n, X^{-1}, X)$ such that for $j \leq l$,

$$\left\|\Delta_j(f_l \circ X^{-1})\right\|_\infty \leq C2^{\gamma j}\zeta(l) \tag{A.14}$$

To show the applicability of this result to proposition A.0.5, set

$$
\begin{aligned}
\tilde{\zeta}(m,t) &= 2^{-\mu(m,t)}, \\
\tilde{\sigma}(m,t) &= 2^{-\eta(m,t)}.
\end{aligned}
$$

By the concavity of $m \mapsto \mu(m,t)$, $\log_2 \tilde{\zeta}(\cdot, t)$ is convex. From (A.7), one has that for sufficiently small m, $\tilde{\zeta}(m,t) = e^{C_0 t}2^{-m} = \tilde{\sigma}(m, T-t)$. With this in mind, I set

$$\zeta(m,t) = \frac{1}{4}e^{C_0 t}\tilde{\zeta}(m-2,t) \tag{A.15}$$

$$C_{X(t)} = 4e^{C_0 t}. \tag{A.16}$$

Then by proposition A.0.6 together with (A.11), I am now in a position to prove proposition A.0.5:

Proof of Proposition A.0.5. For part (1), Fix $N \geq 1$. Using propositions A.0.4 and A.0.7, one has:

$$\begin{aligned}
\sum_{j=-1}^{N} \left\| \Delta_j (f \circ X^{-1}(t)) \right\|_\infty &\leq \sum_{j=-1}^{N} \sum_{l=-1}^{\infty} \left\| \Delta_j (f_l \circ X^{-1}(t)) \right\|_\infty \\
&\leq \sum_{j=-1}^{N} \left(\sum_{l=-1}^{m} + \sum_{l=m+1}^{\infty} \right) \left\| \Delta_j (f_l \circ X^{-1}(t)) \right\|_\infty \\
&\leq C(N+2) \sum_{l=-1}^{m} \left\| \Delta_j f_l \right\|_\infty \\
&\qquad + \sum_{j=-1}^{N} C 2^{\gamma j} \sum_{l=m+1}^{\infty} \zeta(l,t) \sup_{k \in \mathbb{Z}^n} |a_{l,k}| \\
&\leq C(N+2) \Gamma(m) \left\| f \right\|_\Gamma \\
&\qquad + C 2^{\gamma N} \sum_{l=m+1}^{\infty} \zeta(l,t) \sup_{k \in \mathbb{Z}^n} |a_{l,k}|,
\end{aligned} \tag{A.17}$$

where m will be determined later. Note that I have made use of the fact that the constant in proposition A.0.7 is uniformly bounded for $t \in [0,T]$. To handle the second sum, I use an Abel's lemma argument similar to (3.50), and find, for $d_m = \sum_{l=-1}^{m} \sup_{k \in \mathbb{Z}^n} |a_{l,k}|$, that

$$\begin{aligned}
\sum_{l=m+1}^{\infty} \zeta(l,t) \sup_{k \in \mathbb{Z}^n} |a_{l,k}| &= \sum_{l=m+1}^{\infty} \zeta(l,t)(d_l - d_{l-1}) \\
&\leq -d_m \zeta(m+1,t) + \left\| f \right\|_\Gamma \int_{m+1}^{\infty} -\partial_\xi \zeta(\xi,t) \Gamma(\xi) \mathrm{d}\xi.
\end{aligned} \tag{A.18}$$

Next, observe that by property (ii) of $\Gamma(\xi)$ and the definition of $\zeta(\xi)$,

$$\begin{aligned}
\int_{m+1}^{\infty} -\partial_\xi \zeta(\xi,t) \Gamma(\xi) \mathrm{d}\xi &\leq C \int_{m+1}^{\infty} \partial_\xi \mu(\xi-2,t) 2^{-\mu(\xi-2,t)} \Gamma(\xi) \mathrm{d}\xi \\
&= C \int_{m-1}^{\infty} \partial_\xi \mu(\xi,t) 2^{-\mu(\xi,t)} \Gamma(\xi+2) \mathrm{d}\xi
\end{aligned} \tag{A.19}$$

$$
\begin{aligned}
&\leq C \int_{m-1}^{\infty} \partial_\xi \mu(\xi,t) 2^{-\mu(\xi,t)} \Gamma(\xi) \mathrm{d}\xi \\
&= C \int_{\mu(m-1,t)}^{\infty} 2^{-\mu} \Gamma(\xi(\mu,t)) \mathrm{d}\mu.
\end{aligned}
$$

It is at this point that I make use of assumption (A.2):

$$
\begin{aligned}
\Gamma(\xi(\mu,t)) &= \Gamma(\mu) + \int_0^t \Gamma'(\mu(\tau))\dot{\mu}(\tau)\mathrm{d}\tau \\
&\leq \Gamma(\mu) + C_0(\log_2 e) \int_0^t \lambda(\tau)\Gamma'(\mu(\tau))\tilde{\Gamma}_1(\mu(\tau),\tau)\mathrm{d}\tau \\
&\leq \Gamma(\mu) + C(\log_2 e) \int_0^t \lambda(\tau)\Gamma(\mu(\tau))\mathrm{d}\tau.
\end{aligned} \tag{A.20}
$$

By Gronwall's inequality, this gives

$$
\Gamma(\xi(\mu,t)) \leq \Gamma(\mu) 2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau}. \tag{A.21}
$$

Combined with (A.18), (A.19), one has:

$$
\begin{aligned}
\sum_{l=m+1}^{\infty} \zeta(l,t) \sup_{k\in\mathbb{Z}^n} |a_{l,k}| &\leq C \|f\|_\Gamma 2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau} \int_{\mu(m-1,t)}^{\infty} 2^{-\mu}\Gamma(\mu)\mathrm{d}\mu \\
&\leq C \|f\|_\Gamma 2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau} \Gamma(m) 2^{-\mu(m-1,t)} \\
&\leq C \|f\|_\Gamma 2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau} \Gamma(m)\zeta(m+1,t).
\end{aligned}
$$

Note that my choice of m in (A.17) has been arbitrary until this point. By definition of μ, I can choose m such that

$$
\mu(m-1,t) \leq \gamma N \leq \mu(m-2,t), \tag{A.22}
$$

which implies that

$$
\sum_{l=m+1}^{\infty} \zeta(l,t) \sup_{k\in\mathbb{Z}^n} |a_{l,k}| \leq C \|f\|_\Gamma 2^{C\gamma N\int_0^t \lambda(\tau)\mathrm{d}\tau} \Gamma(m). \tag{A.23}
$$

To bound $\Gamma(m)$ I use another Gronwall-type argument along with (A.9) to write

$$\begin{aligned}
\Gamma(\mu) - \Gamma(m) &= \int_0^t \Gamma'(\mu(\tau))\dot{\mu}(\tau)\mathrm{d}\tau \\
&\leq -C_0(\log_2 e)\int_0^t \lambda(\tau)\Gamma'(\mu(\tau))\tilde{\Gamma}_1(\mu(\tau),\tau)\mathrm{d}\tau \\
&\leq -C\int_0^t \lambda(\tau)\Gamma(\mu(\tau))\mathrm{d}\tau
\end{aligned}$$

which implies that

$$\Gamma(m) \leq \Gamma(\mu(m,t))2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau}. \tag{A.24}$$

Finally, observe that by construction, $\partial_m\mu(m,t) \leq 1$ a.e. m, which gives, by my choice of m,

$$\begin{aligned}
\Gamma(m) &\leq C\Gamma(\gamma N)2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau} \\
&\leq C\Gamma(N)2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau}
\end{aligned}$$

where I use property (ii) of $\Gamma(\xi)$ for the second inequality. Combined with (A.17) and (A.21), I conclude that

$$\sum_{j=-1}^{N} \left\|\Delta_j(f\circ X^{-1}(t))\right\|_\infty \leq C\Gamma_1(N)\left\|f\right\|_\Gamma 2^{C\int_0^t \lambda(\tau)\mathrm{d}\tau}, \tag{A.25}$$

as desired.

To prove the second assertion of proposition A.0.5, I return to (A.20), and use the stronger assumption on $\Gamma(\xi)$, $\Gamma_1(\xi)$ to write

$$\Gamma(\xi(\mu,t)) = \Gamma(\mu) + C\int_0^t \lambda(\tau)\mathrm{d}\tau, \tag{A.26}$$

which transforms (A.19) into

$$\int_{m+1}^{\infty} -\partial_\xi \zeta(\xi,t)\Gamma(\xi)\mathrm{d}\xi \le C\zeta(m+1,t)\left[\Gamma(\mu(m,t)) + \int_0^t \lambda(\tau)\mathrm{d}\tau\right]. \quad \text{(A.27)}$$

This, in turn, gives

$$\sum_{l=m+1}^{\infty} \zeta(l,t) \sup_{k\in\mathbb{Z}^n} |a_{l,k}| \le C\left\|f\right\|_\Gamma \zeta(m+1,t)\left[\Gamma(\mu(m,t)) + \int_0^t \lambda(\tau)\mathrm{d}\tau\right].$$

Choosing m such that (A.22) is satisfied, the second term on the right hand side of (A.17) is then bounded by

$$\begin{aligned} 2^{\gamma N} \sum_{l=m+1}^{\infty} \zeta(l,t) \sup_{k\in\mathbb{Z}^n} |a_{l,k}| &\le C\left\|f\right\|_\Gamma \left[\Gamma(\mu(m,t)) + \int_0^t \lambda(\tau)\right] \quad \text{(A.28)} \\ &\le C\left\|f\right\|_\Gamma \left[\Gamma(N) + \int_0^t \lambda(\tau)\right]. \end{aligned}$$

From (A.17) and (A.28), one has that for $N \ge 1$,

$$\sum_{j=-1}^{N} \left\|\Delta_j(f\circ X^{-1}(t))\right\|_\infty \le C\left\|f\right\|_\Gamma \Gamma_1(N)\left(1 + \int_0^t \lambda(\tau)\mathrm{d}\tau\right),$$

which yields the desired bound on $\|f\circ X^{-1}(t)\|_{\Gamma_1}$. □

Bibliography

[1] H. Bahouri and J.-Y. Chemin. Équations de transport relatives á des champs de vecteurs non-lipschitziens et mécanique des fluides. *Arch. Rational Mech. Anal.*, 127(2):159–181, 1994.

[2] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[3] J.-M. Bony. Calcul symbolique et singularités des solutions des équations aux dérivées partielles non linéaires. In *Journées: Équations aux Dérivées Partielles (Saint-Cast, 1979)*, pages Exp. No. 1, 7. École Polytech., Palaiseau, 1979.

[4] J. V. Boussinesq. *Théorie analytique de la chaleur*, volume 2. Gauthier-Villars, Paris, 1903.

[5] J. R. Cannon and Emmanuele DiBenedetto. The initial value problem for the Boussinesq equations with data in L^p. In *Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979)*, volume 771 of *Lecture Notes in Math.*, pages 129–144. Springer, Berlin, 1980.

[6] Dongho Chae. Global regularity for the 2D Boussinesq equations with partial viscosity terms. *Adv. Math.*, 203(2):497–513, 2006.

[7] Dongho Chae, Sung-Ki Kim, and Hee-Seok Nam. Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. *Nagoya Math. J.*, 155:55–80, 1999.

[8] Dongho Chae and Hee-Seok Nam. Local existence and blow-up criterion for the Boussinesq equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 127(5):935–946, 1997.

[9] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[10] Jean-Yves Chemin. Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. *J. Anal. Math.*, 77:27–50, 1999.

[11] Elaine Cozzi. *Incompressible Fluids with Vorticity in Besov Spaces*. PhD thesis, University of Texas at Austin, 2007.

[12] Elaine Cozzi and James P. Kelliher. Vanishing viscosity in the plane for vorticity in borderline spaces of Besov type. *J. Differential Equations*, 235(2):647–657, 2007.

[13] Raphaël Danchin and Marius Paicu. Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data. *Comm. Math. Phys.*, 290(1):1–14, 2009.

[14] Weinan E and Chi-Wang Shu. Small-scale structures in Boussinesq convection. *Phys. Fluids*, 6(1):49–58, 1994.

[15] Michael Frazier and Björn Jawerth. Decomposition of Besov spaces. *Indiana Univ. Math. J.*, 34(4):777–799, 1985.

[16] Bo Ling Guo. Spectral method for solving two-dimensional Newton-Boussinesq equations. *Acta Math. Appl. Sinica (English Ser.)*, 5(3):208–218, 1989.

[17] Taoufik Hmidi and Sahbi Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. Ration. Mech. Anal.*, 189(2):283–300, 2008.

[18] Pijush K. Kundu and Ira M. Cohen. *Fluid Mechanics, 3rd edition*. Elsevier Academic Press, New York, 2004.

[19] Xiaofeng Liu, Meng Wang, and Zhifei Zhang. Local well-posedness and blowup criterion of the Boussinesq equations in critical Besov spaces. *J. Math. Fluid Mech.*, 12(2):280–292, 2010.

[20] Andrew Majda. *Introduction to PDEs and waves for the atmosphere and ocean*, volume 9 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[21] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[22] Yves Meyer. Remarques sur un théorème de J.-M. Bony. In *Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980)*, number suppl. 1 in Rendiconti del Circolo Matematico di Palermo. Serie II, pages 1–20, 1981.

[23] Hee Chul Pak and Young Ja Park. Existence of solution for the Euler equations in a critical Besov space $\mathbf{B}^1_{\infty,1}(\mathbb{R}^n)$. *Comm. Partial Differential Equations*, 29(7-8):1149–1166, 2004.

[24] Jaak Peetre. *New thoughts on Besov spaces.* Mathematics Department, Duke University, Durham, N.C., 1976. Duke University Mathematics Series, No. 1.

[25] Alain Pumir and Eric D. Siggia. Development of singular solutions to the axisymmetric Euler equations. *Phys. Fluids A*, 4(7):1472–1491, 1992.

[26] E. A. Spiegel and G. Veronis. On the Boussinesq approximation for a compressible fluid. *Astrophys. J.*, 131:442–447, 1960.

[27] Yasushi Taniuchi. Remarks on global solvability of 2-D Boussinesq equations with non-decaying initial data. *Funkcial. Ekvac.*, 49(1):39–57, 2006.

[28] Hans Triebel. *The structure of functions*, volume 97 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[29] M. R. Ukhovskii and V. I. Yudovich. On the equations of steady-state convection. *J. Appl. Math. Mech.*, 27:432–440, 1963.

[30] Misha Vishik. Hydrodynamics in Besov spaces. *Arch. Ration. Mech. Anal.*, 145(3):197–214, 1998.

[31] Misha Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. Sci. École Norm. Sup. (4)*, 32(6):769–812, 1999.

[32] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066, 1963.

[33] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.*, 2(1):27–38, 1995.

Vita

Jacob Benjamin Glenn-Levin was born in San Diego, California. He attended Torrey Pines High School in Del Mar, California and graduated with honors in 1997. Jacob earned a dual Bachelor of Arts degree in Mathematics and Rhetoric from the University of California at Berkeley in May 2002. He began his graduate studies at the University of Texas at Austin in August 2005, earning his M.A. in May 2011, and will earn his doctorate almost exactly a decade after he earned his B.A.

Permanent address: jacobgl@gmail.com

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.